Linear Programming Problem (L.P.P.)

Graphical sol^n

Problem :-

1) Solve the following L.P.P. graphically :

Minimize $z = 2x_1 - x_2$

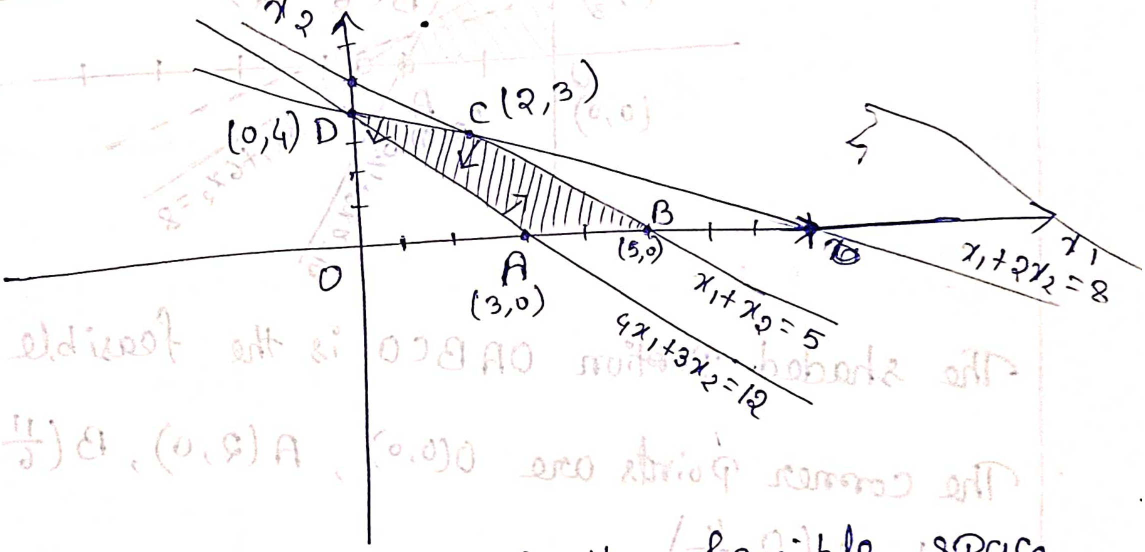
subject to $x_1 + x_2 \leq 5$

$x_1 + 2x_2 \leq 8$

$4x_1 + 3x_2 \geq 12$

$x_1, x_2 \geq 0$.

$\Rightarrow$



The shaded Portion ABCDA is the feasible space.

The Corner Points are A, B, C and D.

The Co-ordinates of the corner Points are A(3,0), B(5,0), C(2,3) and D(0,4).

The values of the objective function at the corner Points are

$z_A = 6$, $z_B = 10$, $z_C = 1$ and $z_D = -4^*$

The optimality occurs at the corner Point D.

$\therefore$ The optimal solution is

$x_1 = 0$, $x_2 = 4$ and $z_{min} = -4$

2) Solve the following LPP Graphically:

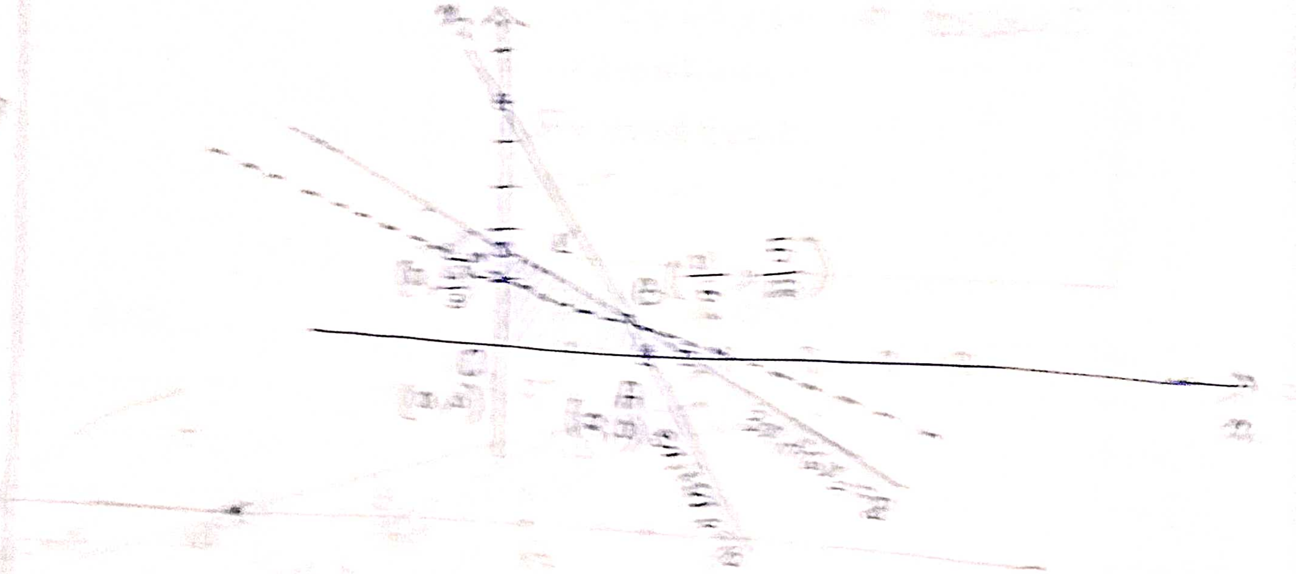
Maximise $z = [illegible]$

Subject to

$[illegible] \le [illegible]$

$[illegible] \le 10$

$x_1, x_2 \ge 0$



The shaded portion OABCO is the feasible region.

The corner points are O(0,0), A([illegible]), B([illegible]) and C([illegible])

The values of the objective function at the corner points are

$z_O = 0$, $z_A = [illegible]$, $z_B = [illegible]$, $z_C = [illegible]$

The maximality occurs at the corner point [illegible]

The optimal solution is

$x_1 = [illegible]$, $x_2 = [illegible]$, $z_{max} = [illegible]$

[Note :- Profit line :- The profit line corresponding to $z=3$ is $x_1+3x_2=3$ which is shown in the graph as dotted line.

Since, the problem is maximizing, we replace the profit line parallely away from origin and we observe that, at a time, profit line passes through the corner point C.

∴ Optimality occurs at the corner point C.

∴ Optimal solution is $x_1=0$, $x_2=\frac{4}{3}$ and $z_{max}=4$]

3) Solve the following problem graphically :

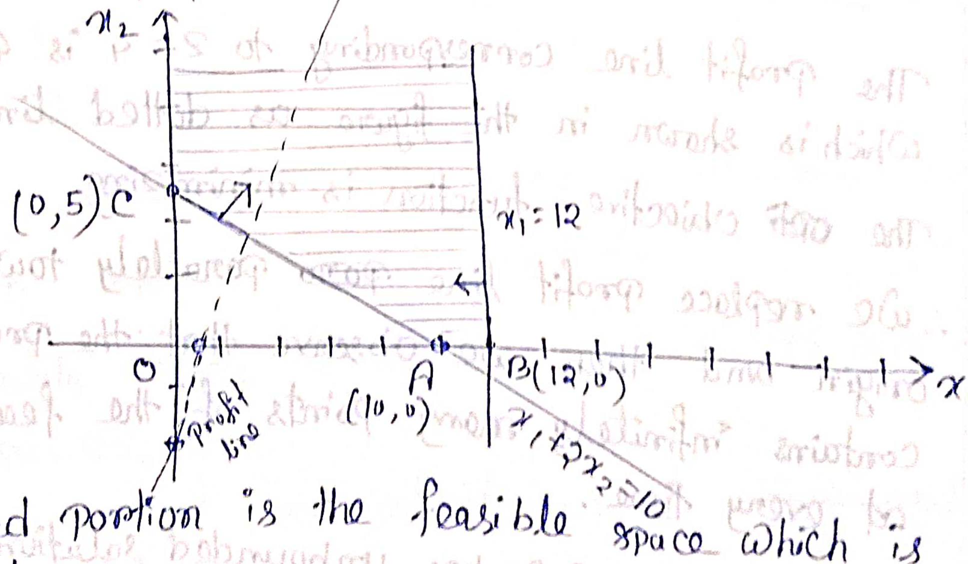
max $z=4x_1-x_2$

Subject to $x_1+2x_2\geq 10$

$x_1\leq 12$

$x_1, x_2\geq 0$

⇒



The shaded portion is the feasible space which is unbounded.

The profit line corresponding to $z=4$ is $4x_1-x_2=4$ which is shown in the figure as dotted line.

Here, the objective function is maximizing.

We replace the profit line parallely away from origin and we observe that the profit line contains exactly one point of the feasible space, namely $B(12,0)$.

∴ Optimality occurs at corner point B.

∴ optimal solution is $x_1 = 12$, $x_2 = 0$ and $z_{max} = 48$.

4) Solve the following L.P.P. graphically:

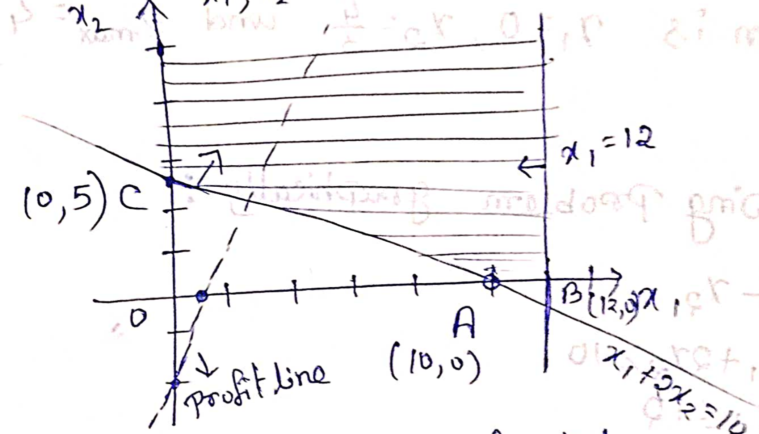
$\min z = 4x_1 - x_2$

subject to $x_1 + 2x_2 \geq 10$

$x_1 \leq 12$

$x_1, x_2 \geq 0$,

=>



The shaded portion is the feasible space which is unbound

The profit line corresponding to $z = 4$ is $4x_1 - x_2 = 4$ which is shown in the figure as dotted line.

The objective function is minimizing.

∴ We replace profit line parallely towards the origin and then we observe that the profit line contains infinitely many points of the feasible space at every time.

∴ The given L.P.P. has unbounded solution.

Convex set

(i) **Point set :-** A set is said to be point set if it's elements are points in E^n.

For example,

$X_1 = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$ is a point set in E^2 ~~whose~~ which represents a circular disc in E^2 whose radius is 1 unit and centre at origin.

(ii) **Hyperplane :-** In E^n, the set $H = \{x : cx = z\}$ is known as hyperplane, where $c = (c_1, c_2, \ldots, c_n)$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$. i.e. in E^n, the eqn of Hyperplane is $c_1x_1 + c_2x_2 + \ldots + c_nx_n = z$.

In E^2, a hyperplane is a straight line and in E^3, a hyperplane is a plane.

[**Note :-** $c = (c_1, c_2, \ldots, c_n)$ is called the normal to the hyperplane.]

(iii) **Open and closed half spaces :-**

Consider, $X_1 = \{x : cx > z\}$

$X_2 = \{x : cx = z\}$

$X_3 = \{x : cx < z\}$

X_1 and X_3 are called open half spaces.

The half spaces $cx \geq z$ and $cx \leq z$ are called closed half spaces.

Hypersphere :– The eqⁿ $|x-a| = \epsilon$ is called ~~to~~ the hypersphere in E^n.

In set builder form,

$$H = \{ x : |x-a| = \epsilon \} \text{ is a hypersphere.}$$

In E^2, a hypersphere is a circle and in E^3, a hypersphere is a sphere.

Neighbourhood of a Point :– The ϵ-nbd of a Point a in E^n is the set $X = \{ x : |x-a| < \epsilon \}$

Interior Point :– A Point a of a set X is said to be interior point if ~~all the points of nbd of a are the points of~~ $\exists$ an nbd of a whose Points are Points of X.

Open set :– A set X is said to be open set if all it's Points are interior point.

e.g. $X = \{ (x_1, x_2) : x_1^2 + x_2^2 < 1 \}$ is an open set.

Boundary Points :– A Point w of a set X is said to be boundary Point of X if every nbd of w contains Points of X and Points not of X.

For example, the boundary Points of $X = \{ (x_1, x_2) : x_1^2 + x_2^2 \leq 1 \}$ are the Points on the circumference of the circle $x_1^2 + x_2^2 = 1$.

closed set :– A set X is said to be closed if it contains all it's boundary points.

For example, $X = \{ (x_1, x_2) : x_1^2 + x_2^2 \leq 4 \}$ is a closed set.

Bounded set :- A set X is said to be bounded if $\exists$ a +ve real number r such that $|x| < r, \ \forall x \in X$.

Line :- The eqn of the line joining the points x_1 and x_2 in E^n is $x = \lambda x_1 + (1-\lambda) x_2$, λ being real.

In set form, $X = \{x : x = \lambda x_1 + (1-\lambda) x_2\}$ represents a line.

The eqn of line segment joining the points x_1 and x_2 in E^n is $x = \lambda x_1 + (1-\lambda) x_2$, $0 \le \lambda \le 1$

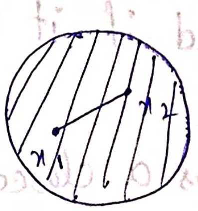
Convex Combination :- The point x is said to be the convex combination of the points $x_1, x_2, \ldots, x_n$ if $\exists$ real numbers $\lambda_i \ge 0, i = 1, 2, \ldots, n$ such that

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$$

where $\sum_{i=1}^{n} \lambda_i = 1$.

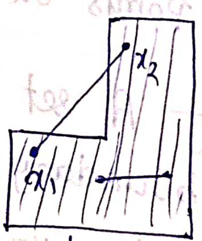
The point $x = \lambda x_1 + (1-\lambda) x_2$, $0 \le \lambda \le 1$ is the convex combination of the points x_1 and x_2 which is the line segment joining x_1 and x_2.

Convex set :- A set X is said to be convex set if for any $x_1, x_2 \in X$, $x = \lambda x_1 + (1-\lambda) x_2 \in X$, where $0 \le \lambda \le 1$.

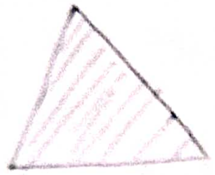


Convex

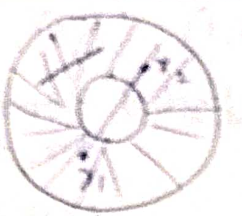
Convex



not Convex



Convex



not Convex

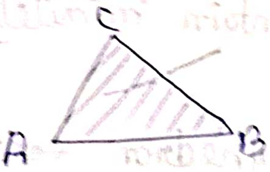
not Convex

Extreme Point :- Let, X be a convex set. A point x is said to be extreme point of X if there is no two points x_1 and x_2 other from x, ~~the line segment~~ ~~of the~~ such that x lies on the line segment joining x_1 and x_2.

Example, 1) The extreme point of the set ~~$X=\{(x_1,x_2)$~~ $X=\{(x_1,x_2): x_1^2+x_2^2 \le 1\}$ are the points on the circumference of the circle $x_1^2+x_2^2=1$.

2) 

A, B, C and D are extreme points.

3) 

A, B and C are extreme points.

4) The set $X=\{(x_1,x_2): x_1+2x_2=4\}$ has no extreme point.

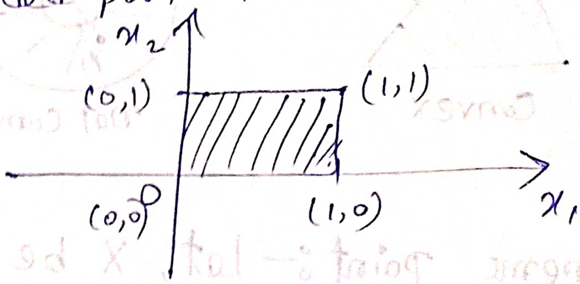
Convex hull :- The convex hull of a set X ~~is~~ contains all the convex combination of points of X and is denoted by $C(X)$.

For example,

1) if $X=\{(x_1,x_2): x_1^2+x_2^2=1\}$

then $C(X)=\{(x_1,x_2): x_1^2+x_2^2 \le 1\}$

2) If $X = \{(1,0), (0,1), (1,1), (0,0)\}$ then $c(X)$ is the shaded portion.



3) The convex hull of a set containing the vertic of a cube is the solid cube.

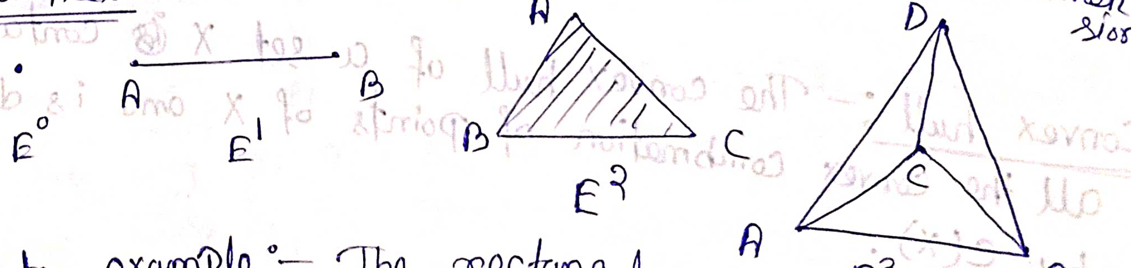
Convex Polyhedron :- If X be a finite set, then $C(X)$ is called convex Polyhedron.

Example :- If X contain 8 vertices of a cube, then it's convex hull is a convex polyhedron.

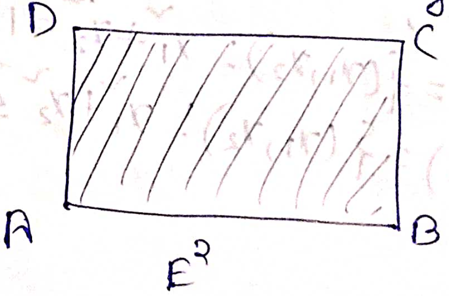
Counter example :- Let, $X = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$, then $c(X) = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$ is not a convex Polyhedron, Since, X contain infinitely many points.

Simplex :- A convex Polyhedron ~~containing~~ in E^n having exactly $(n+1)$ vertices is called simplex.

Examples :- The following are simplex in respective dimension



Counter example :- The rectangular region is not a simplex in E^2.



Basic Feasible Solution (B.F.S.)

Problem:-

1) Find all the basic solutions of the system

$$2x_1 + x_2 + 4x_3 = 11$$
$$3x_1 + x_2 + 5x_3 = 14$$

⇒ The given system of eq^n is $\left.\begin{array}{l} 2x_1 + x_2 + 4x_3 = 11 \\ 3x_1 + x_2 + 5x_3 = 14 \end{array}\right\} \text{---(i)}$

Here, the no. of variable is $n = 3$ and the number of eq^{n}s is $m = 2$.

∴ The maximum no. of basic solution is $^nc_m = {}^3c_2 = 3$.

Here, $n - m = 3 - 2 = 1$

(1) Choosing x_1 as non-basic variable, i.e. $x_1 = 0$, the system (i) reduced to

$$\left.\begin{array}{l} x_2 + 4x_3 = 11 \\ x_2 + 5x_3 = 14 \end{array}\right\} \text{---(ii)}$$

Since, $\begin{vmatrix} 1 & 4 \\ 1 & 5 \end{vmatrix} = 1 \neq 0$, basic solution exists in this case.

Solving the system (ii), we have,

$$x_2 = -1,\ x_3 = 3$$

∴ a basic solution is $(0, -1, 3)$

(2) Choosing x_2 as non-basic variable, i.e. $x_2 = 0$, the system (i) reduced to

$$\left.\begin{array}{l} 2x_1 + 4x_3 = 11 \\ 3x_1 + 5x_3 = 14 \end{array}\right\} \text{---(iii)}$$

Since, $\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = -2 \neq 0$

∴ basic solution exists in this case.

Solving the system (iii), we have,

$$x_1 = \frac{1}{2},\ x_3 = \frac{5}{2}$$

∴ Basic solution is $\left(\frac{1}{2}, 0, \frac{5}{2}\right)$

(3) Choosing x_3 as non-basic variable, i.e. $x_3 = 0$, the system (i) reduced to

$$\left.\begin{array}{l} 2x_1 + x_2 = 11 \\ 3x_1 + x_2 = 14 \end{array}\right\} \text{—(iv)}$$

Since, $\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -1 \neq 0$, basic solution exists in this case.

solving the system (iv), we have,

$$x_1 = 3, \; x_2 = 5$$

$\therefore$ another basic solution is $(3, 5, 0)$

$\therefore$ The basic solution of the given system are $(0, -1, 3)$, $\left(\frac{1}{2}, 0, \frac{5}{2}\right)$ and $(3, 5, 0)$.

[Note:— Among the three B.S., $(3, 5, 0)$ and $\left(\frac{1}{2}, 0, \frac{5}{2}\right)$ are B.F.S. since for these two solutions, all the basic variables ≥ 0.

On the other hand, the B.S. $(0, -1, 3)$ is not B.F.S.]

2) Find all the B.S. of the system $\begin{array}{l} x_1 + 2x_3 = 1 \\ x_2 + x_3 = 4 \end{array}$. Discuss the nature of the B.S.

$\Rightarrow$ The given system is $\left.\begin{array}{l} x_1 + 2x_3 = 1 \\ x_2 + x_3 = 4 \end{array}\right\}$ —(1)

Here, $n = 3$, $m = 2$

$\therefore$ maximum no. of B.S. is ${}^3C_2 = 3$.

Here, $n - m = 3 - 2 = 1$.

(i) choosing x_1 as non-basic variable, the system (i) reduced to, $\left.\begin{array}{l} 2x_3 = 1 \\ x_2 + x_3 = 4 \end{array}\right\}$ —(2)

Since, $\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} = -2 \neq 0$, B.S. exists in this case.

Solving (2) we have,

$$x_3 = \frac{1}{2}, \quad x_2 = \frac{7}{2}$$

$\therefore$ Basic solution is $\left(0, \frac{7}{2}, \frac{1}{2}\right)$

(ii) choosing x_2 as non-basic variable, the system (1) reduced to

$$\left.\begin{array}{l} x_1 + 2x_3 = 1 \\ x_3 = 4 \end{array}\right\} \text{—(3)}$$

Since,

$\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$, B.S. exists in this case.

Solving (3), we have,

$$x_3 = 4, \quad x_1 = -7$$

$\therefore$ Basic solution is $(-7, 0, 4)$

(iii) choosing x_3 as non-basic variable, the system (1) reduced to

$$\left.\begin{array}{l} x_1 = 1 \\ x_2 = 4 \end{array}\right\} \text{—(4)}$$

Since, $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$, B.S. exists in this case.

~~$\therefore$ The B.~~

$\therefore$ Another B.S. is $(1, 4, 0)$

Among the Basic solutions,

$\left(0, \frac{7}{2}, \frac{1}{2}\right)$ and $(1, 4, 0)$ are B.F.S.

$(-7, 0, 4)$ is a B.S. but not feasible.

All the Basic solutions are non-degenerate.

3) Find all the basic feasible solution of the eq^{n}s
$2x_1+6x_2+2x_3+x_4=3$ Show that all the B.F.S. are
$6x_1+4x_2+4x_3+6x_4=2$
degenerate.

⇒ The given system is $\left.\begin{array}{l}2x_1+6x_2+2x_3+x_4=3\\ 6x_1+4x_2+4x_3+6x_4=2\end{array}\right\}$ —(i)

Here, $n=4$, $m=2$

∴ The maximum no. of basic solution is ${}^nC_m = {}^4C_2 = 6$

To get basic sol^n, $n-m=\cancel{8}4-2=2$ variables are assigned to be zero.

(1) Choosing x_1 and x_2 as non-basic variables, i.e. $x_1=x_2=0$,
∴ The system (i) reduced to $\left.\begin{array}{l}2x_3+x_4=3\\ 4x_3+6x_4=2\end{array}\right\}$ —(ii)

Since, $\begin{vmatrix}2 & 1\\ 4 & 6\end{vmatrix}=8\neq 0$, basic sol^n exists in this case.

solving (ii), we have, $x_3=2$, $x_4=-1$

∴ ~~The~~ A basic sol^n ~~is~~ is $(0,0,2,-1)$

(2) Choosing x_1 and x_3 as non-basic variables, i.e. $x_1=x_3=0$,
the system (i) reduced to, $\left.\begin{array}{l}6x_2+x_4=3\\ 4x_2+6x_4=2\end{array}\right\}$ —(iii)

Since, $\begin{vmatrix}6 & 1\\ 4 & 6\end{vmatrix}=32\neq 0$, basic sol^n exists in this case.

solving system (iii) we have,
$x_2=\frac{1}{2}$, $x_4=0$

∴ Another basic sol^n is $(0,\frac{1}{2},0,0)$

(3) Choosing x_1 and x_4 as non-basic variables, i.e. $x_1=x_4=0$,
the system (i) reduced to, $\left.\begin{array}{l}6x_2+2x_3=3\\ 4x_2+4x_3=2\end{array}\right\}$ —(iv)

Since, $\begin{vmatrix}6 & 2\\ 4 & 4\end{vmatrix}=16\neq 0$, basic sol^n exists in this case.

solving the system (iv),
$x_2=\frac{1}{2}$, $x_3=0$

∴ Another basic solution is $(0,\frac{1}{2},0,0)$

$x_1 = 3 + \frac{7}{3}$
$= \frac{16}{3}$

(4) Choosing x_2, x_3 as non-basic variables, i.e. $x_2 = x_3 = 0$, system (i) reduced to,

$$\left.\begin{array}{l} 2x_1 + x_4 = 3 \\ 6x_1 + 6x_4 = 2 \end{array}\right\} \text{—(v)}$$

Since, $\begin{vmatrix} 2 & 1 \\ 6 & 6 \end{vmatrix} = 6 \neq 0$, basic solution exists in this case.

Solving system (v),

$$x_1 = \frac{8}{3}, \quad x_4 = -\frac{7}{3}$$

∴ Another basic solution is, $\left(\frac{8}{3}, 0, 0, -\frac{7}{3}\right)$

(5) Choosing x_2, x_4 as non-basic variables, i.e. $x_2 = x_4 = 0$, system (i) reduced to,

$$\left.\begin{array}{l} 2x_1 + 2x_3 = 3 \\ 6x_1 + 4x_3 = 2 \end{array}\right\} \text{—(vi)}$$

Since, $\begin{vmatrix} 2 & 2 \\ 6 & 4 \end{vmatrix} = -4 \neq 0$, basic solution exists in this case.

Solving system (vi), $x_1 = -2$, $x_3 = \frac{7}{2}$

∴ Another basic solution is, $\left(-2, 0, \frac{7}{2}, 0\right)$

(6) Choosing x_3, x_4 as non-basic variables, i.e. $x_3 = x_4 = 0$, system (i) reduced to,

$$\left.\begin{array}{l} 2x_1 + 6x_2 = 3 \\ 6x_1 + 4x_2 = 2 \end{array}\right\} \text{—(vii)}$$

∵ $\begin{vmatrix} 2 & 6 \\ 6 & 4 \end{vmatrix} = -28 \neq 0$, basic sol^n exists in this case,

Solving (vii),

$$x_1 = 0, \quad x_2 = \frac{1}{2}$$

∴ Another basic sol^n is $\left(0, \frac{1}{2}, 0, 0\right)$

∴ The basic feasible sol^n of the system is $\left(0, \frac{1}{2}, 0, 0\right)$

Clearly, among all the B.F.S. one basic variables is zero, hence all of them are degenerate.

Reduction of B.F.S. from Feasible sol^n

Problem :-

1) (2, 3, 1) is a feasible sol^n of the system

$2x_1+x_2+4x_3=11$

$3x_1+x_2+5x_3=14$

Reduce the feasible sol^n to a B.F.S.

⇒ The given system is

$$\left.\begin{array}{l} 2x_1+x_2+4x_3=11 \\ 3x_1+x_2+5x_3=14 \end{array}\right\} \text{—(i)}$$

The given F.S. is $x_1=2$, $x_2=3$, $x_3=1$

The characteristic eq^n is

$$\left.\begin{array}{l} 2\lambda_1+\lambda_2+4\lambda_3=0 \\ 3\lambda_1+\lambda_2+5\lambda_3=0 \end{array}\right\} \text{—(ii)}$$

Solving (ii) by means of cross-multiplication, we have,

$$\frac{\lambda_1}{1}=\frac{\lambda_2}{2}=\frac{\lambda_3}{-1}=k \text{ (say)}$$

∴ $\lambda_1=k$, $\lambda_2=2k$, $\lambda_3=-k$

taking $k=1$, we have, $\lambda_1=1$, $\lambda_2=2$, $\lambda_3=-1$.

Now, $\min_j\left(\frac{x_j}{\lambda_j}; \lambda_j>0\right)=\min\left(\frac{2}{1},\frac{3}{2}\right)$

$=\frac{3}{2}=\frac{x_2}{\lambda_2}\left(\frac{x_r}{\lambda_r}\right)$

$r=2$

New x_j = old $x_j-\frac{x_r}{\lambda_r}\lambda_j$, $j=1,2,3$

∴ New $x_1=2-\frac{3}{2}\cdot1=\frac{1}{2}$

New $x_2=3-\frac{3}{2}\cdot2=0$

New $x_3=1-\frac{3}{2}(-1)=\frac{5}{2}$

∴ $\left(\frac{1}{2},0,\frac{5}{2}\right)$ is a basic feasible sol^n.

∴ New x_j = old $x_j - \left(\frac{x_r}{\lambda_r}\right)\lambda_j$, $j = 1, 2, 3$

∴ New x_1 = old $x_1 - \frac{x_r}{\lambda_r}\lambda_1$

$= 2 - 1 \cdot 1 = 1$

New x_2 = old $x_2 - \left(\frac{x_r}{\lambda_r}\right)\lambda_2$

$= 1 - 1 \cdot 1 = 0$

New x_3 = old $x_3 - \left(\frac{x_r}{\lambda_r}\right)\lambda_3$

$= 3 - 1 \cdot 2 = 1$

∴ $(1, 0, 1)$ is the corresponding B.F.S.

3) $x_1 = 1, x_2 = 1, x_3 = 1$ and $x_4 = 0$ is a feasible sol^n of the system $x_1 + 2x_2 + 4x_3 + x_4 = 7$, $2x_1 - x_2 + 3x_3 - 2x_4 = 4$. Reduce the F.S. to two different B.F.S.

⇒ The given system is,

$$\left.\begin{aligned} x_1 + 2x_2 + 4x_3 + x_4 &= 7 \\ 2x_1 - x_2 + 3x_3 - 2x_4 &= 4 \end{aligned}\right\} \text{—(i)}$$

the given F.S. is, $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $x_4 = 0$

The characteristic eq^n is,

$$\left.\begin{aligned} \lambda_1 + 2\lambda_2 + 4\lambda_3 &= 0 \\ 2\lambda_1 - \lambda_2 + 3\lambda_3 &= 0 \end{aligned}\right\} \text{—(ii)}$$

Solving the system (ii) by means of cross-multiplication

$$\frac{\lambda_1}{6+4} = \frac{\lambda_2}{8-3} = \frac{\lambda_3}{-1-4}$$

$$\frac{\lambda_1}{10} = \frac{\lambda_2}{5} = \frac{\lambda_3}{-5}$$

$$\frac{\lambda_1}{2} = \frac{\lambda_2}{1} = \frac{\lambda_3}{-1} = k \text{ (say)}$$

∴ $\lambda_1 = 2k$, $\lambda_2 = k$, $\lambda_3 = -k$

Choosing $k = 1$, we have,

$\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = -1$

Now, $\min\limits_j \left(\frac{x_j}{\lambda_j} ; \lambda_j > 0\right)$

$= \min\left(\frac{1}{2}, \frac{1}{1}\right) = \frac{1}{2} = \frac{x_r}{\lambda_r}$ (say)

$\therefore$ new x_j = old $x_j - \left(\frac{x_r}{\lambda_r}\right)\lambda_j$, $j = 1, 2, 3$

$\therefore$ new x_1 = old $x_1 - \left(\frac{x_r}{\lambda_r}\right)\lambda_1 = 1 - \frac{1}{2} \cdot 2 = 0$

$\therefore$ new x_2 = old $x_2 - \left(\frac{x_r}{\lambda_r}\right)\lambda_2 = 1 - \frac{1}{2} \cdot 1 = \frac{1}{2}$

$\therefore$ new x_3 = old $x_3 - \left(\frac{x_r}{\lambda_r}\right)\lambda_3 = 1 - \frac{1}{2}(-1) = \frac{3}{2}$

$\therefore$ ~~the~~ one corresponding B.F.S. is $\left(0, \frac{1}{2}, \frac{3}{2}, 0\right)$

Again, $\max\left(\frac{x_j}{\lambda_j} : \lambda_j < 0\right) = \max\left(\frac{1}{-1}\right) = -1 = \frac{x_r}{\lambda_r}$ (say)

$\therefore$ New x_j = old $x_j - \left(\frac{x_r}{\lambda_r}\right)\lambda_j$, $j = 1, 2, 3$

$\therefore$ new x_1 = old $x_1 - \left(\frac{x_r}{\lambda_r}\right)\lambda_1 = 1 - (-1)\,2 = 3$

new x_2 = old $x_2 - \left(\frac{x_r}{\lambda_r}\right)\lambda_2 = 1 - (-1)\,1 = 2$

~~old~~ new x_3 = old $x_3 - \left(\frac{x_r}{\lambda_r}\right)\lambda_3 = 1 - (-1)(-1) = 0$

$\therefore$ ~~another~~ corresponding another B.F.S. is $(3, 2, 0, 0)$

4) $x_1 = 1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 0$ is a feasible sol^n of the set of eq^{n}s $11x_1 + 2x_2 - 9x_3 + 4x_4 = 6$, $15x_1 + 3x_2 - 12x_3 + 5x_4 = 9$. Reduce the corresponding B.F.S.(s) and Prove that one of them is non-degenerate and other is degenerate.

$\Rightarrow$ the given sol^n is, $x_1 = 1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 0$

the given system is,
$$\left.\begin{array}{l} 11x_1 + 2x_2 - 9x_3 + 4x_4 = 6 \\ 15x_1 + 3x_2 - 12x_3 + 5x_4 = 9 \end{array}\right\} \text{—(i)}$$

The characteristic eq^n is,
$$\left.\begin{array}{l} 11\lambda_1 + 2\lambda_2 - 9\lambda_3 = 0 \\ 15\lambda_1 + 3\lambda_2 - 12\lambda_3 = 0 \end{array}\right\} \text{—(ii)}$$

solving the system (ii) by means of cross-multiplication, we have

$$\frac{\lambda_1}{-24 + 27} = \frac{\lambda_2}{-135 + 132} = \frac{\lambda_3}{33 - 30}$$

$$\frac{\lambda_1}{3} = \frac{\lambda_2}{-3} = \frac{\lambda_3}{3}$$

$$\frac{\lambda_1}{1} = \frac{\lambda_2}{-1} = \frac{\lambda_3}{1} = k \text{ (say)}$$

$\therefore \lambda_1 = k$, $\lambda_2 = -k$, $\lambda_3 = k$

Choosing $k = 1$, we have,

$\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 1$

Now, $\min\limits_{j}\left(\frac{x_j}{\eta_j} : \eta_j > 0\right)$

we have, $= \min\left(\frac{1}{1}, \frac{1}{1}\right) = 1 = \frac{x_r}{\eta_r}$ (say)

$\therefore$ New $x_j = \text{old } x_j - \left(\frac{x_r}{\eta_r}\right)\eta_j$, $j = 1, 2, 3$

$\therefore$ new $x_1 = \text{old } x_1 - \left(\frac{x_r}{\eta_r}\right)\eta_1 = 1 - 1\cdot(1) = 0$

new $x_2 = \text{old } x_2 - \left(\frac{x_r}{\eta_r}\right)\eta_2 = 2 - 1\cdot(-1) = 3$

new $x_3 = \text{old } x_3 - \left(\frac{x_r}{\eta_r}\right)\eta_3 = 1 - 1\cdot 1 = 0$

$\therefore$ one corresponding B.F.S. is $(0, 3, 0, 0)$ which is degenerate.

Again, $\max\limits_{j}\left(\frac{x_j}{\eta_j} : \eta_j < 0\right)$

we have, $= \max\left(\frac{2}{-1}\right) = -2 = \frac{x_r}{\eta_r}$ (say)

$\therefore$ New $x_j = \text{Old } x_j - \left(\frac{x_r}{\eta_r}\right)\eta_j$, $j = 1, 2, 3$

$\therefore$ new $x_1 = \text{old } x_1 - \left(\frac{x_r}{\eta_r}\right)\eta_1 = 1 - (-2)\cdot 1 = 3$

new $x_2 = \text{Old } x_2 - \left(\frac{x_r}{\eta_r}\right)\eta_2 = 2 - (-2)(-1) = 0$

new $x_3 = \text{old } x_3 - \left(\frac{x_r}{\eta_r}\right)\eta_3 = 1 - (-2)\,1 = 3$

$\therefore$ Corresponding another B.F.S. is $(3, 0, 3, 0)$ which is non-degenerate.

Homework

1) $x_1 = 2$, $x_2 = 4$, and $x_3 = 1$ is a F.S. to the system of eqⁿs
$$\begin{cases} 2x_1 - x_2 + 2x_3 = 2 \\ x_1 + 4x_2 = 18 \end{cases}$$
Reduce the F.S. to a B.F.S.

(Ans:- $(0, \frac{9}{2}, \frac{13}{4})$)

2) $(1, 1, 2)$ is a F.S. to the system of eqⁿs
$$x_1 + 2x_2 + 3x_3 = 9$$
$$2x_1 - x_2 + x_3 = 3$$
Reduce the F.S. to a B.F.S.

[Ans: $(3, 3, 0)$]

3) Reduce the F.S. $x_1 = 2$, $x_2 = 1$, $x_3 = 1$ of the system of eqⁿs $x_1 + 4x_2 - x_3 = 5$, $2x_1 + 3x_2 + x_3 = 8$ to two different B.F.S.

[Ans:- $(\frac{17}{5}, \frac{2}{5}, 0)$, $(0, \frac{13}{7}, \frac{17}{7})$]

Problem:

1) Solve the following L.P.P. by Simplex method.

$\max z = 5x_1 + 3x_2$

s.t. $3x_1 + 5x_2 \le 15$

$5x_1 + 2x_2 \le 10$

$x_1, x_2 \geq 0$

⇒ The standard form of the given L.P.P. is

$\max z = 5x_1 + 3x_2 + 0 \cdot s_1 + 0 \cdot s_2$

s.t. $3x_1 + 5x_2 + s_1 = 15$

$5x_1 + 2x_2 + s_2 = 10$

$x_1, x_2, s_1, s_2 \geq 0$

s_1, s_2 are slack variables.

Table – I

		C_j	5	3	0	0	
C_B	x_B	b	x_1	x_2	s_1	s_2	
0	s_1	15	3	5	1	0	5
0	s_2	10	~~Key element~~ $\boxed{5}$	2	0	1	2^*
$z_j - c_j$			-5	-3	0	0	
			↑ entering vector			↓ leaving vector	

$z_j - c_j$ ⇒ net evaluation

$\boxed{5}$ ⇒ Key element

Table – II

		C_j	5	3	0	0	
C_B	x_B	b	x_1	x_2	s_1	s_2	
0	s_1	9	0	$\boxed{\frac{19}{5}}$	1	$-\frac{3}{5}$	$\frac{45}{19}$ *
5	x_1	2	1	$\frac{2}{5}$	0	$\frac{1}{5}$	5
$z_j - c_j$			0	-1	0	1	
				↑	↓		

$R_2' = R_2 - \frac{2}{5} \times \frac{5}{19} R_1$

$R_2 - \frac{2}{19} R_1$

$\frac{15}{19} \quad -\frac{10}{19}$

$-\frac{9}{19} + \frac{25}{19}$

$\frac{16}{19}$

Table-III

		C_j	5	3	0	0
C_B	X_B	b	x_1	x_2	s_1	s_2
3	x_2	$\frac{45}{19}$	0	1	$\frac{5}{19}$	$-\frac{3}{19}$
5	x_1	$\frac{20}{19}$	1	0	$-\frac{2}{19}$	$\frac{5}{19}$
$z_j - C_j$			0	0	$\frac{5}{19}$	$\frac{16}{19}$

Since, all $z_j - C_j \geq 0$, ~~the~~ table-III is the optimal table.

$\therefore$ optimal solution is, $x_1 = \frac{20}{19}$, $x_2 = \frac{45}{19}$

and $z_{max} = 5 \times \frac{20}{19} + 3 \times \frac{45}{19} = \frac{235}{19}$

2) Solve the following L.P.P. by simplex method

$\min z = x_1 - 3x_2 + 2x_3$

s.t. $3x_1 - x_2 + 2x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$

$\Rightarrow$ The standard form of the following L.P.P. is

$\max z^* (= -z) = -x_1 + 3x_2 - 2x_3 + 0.s_1 + 0.s_2 + 0.s_3$

s.t. $3x_1 - x_2 + 2x_3 + s_1 = 7$

$-2x_1 + 4x_2 + s_2 = 12$

$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$

$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

Where s_1, s_2, s_3 are slack variables.

Table-1

		C_j	-1	3	-2	0	0	0	
C_B	x_B	b	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	7	3	-1	2	1	0	0	
0	s_2	12	-2	$\boxed{4}$	0	0	1	0	3^*
0	s_2	10	-4	3	8	0	0	1	$\frac{10}{3}$
$z_j - C_j$			1	-3	2	0	0	0	
				↑			↓		

Table-2

		C_j	-1	3	-2	0	0	0	
C_B	x_B	b	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	10	$\boxed{\frac{5}{2}}$	0	2	1	$\frac{1}{4}$	0	4^*
3	x_2	3	$-\frac{1}{2}$	1	0	0	$\frac{1}{4}$	0	
0	s_2	1	$-\frac{5}{2}$	0	8	0	$-\frac{3}{4}$	1	
$z_j - C_j$			$-\frac{1}{2}$	0	2	0	$\frac{3}{4}$	0	
			↑			↓			

Table-3

		C_j	-1	3	-2	0	0	0
C_B	x_B	b	x_1	x_2	x_3	s_1	s_2	s_3
-1	x_1	4	1	0	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0
3	x_2	5	0	1	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	0
0	s_2	11	0	0	10	1	$-\frac{1}{2}$	1
$z_j - C_j$			0	0	$\frac{12}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0

Since, all $z_j - C_j \geq 0$, table – 3 is optimal table.

$\therefore$ Optimal solution, $x_1 = 4$, $x_2 = 5$, $x_3 = 0$

and $z_{min} = -11$

Type - II L.P.P.

(III) Charne's Big - M method / Penalty method :-

Problem :-

1) ~~Max $z = 5x_1 - 2x_2 + 3x_3$~~

~~s.t. $2x_1 + 2x_2 - x_3 \geq$~~

$\max z = 5x_1 + 8x_2$ Solve by Big-M method.

s.t. $3x_1 + 2x_2 \geq 3$

$x_1 + 4x_2 \geq 4$

$x_1 + x_2 \leq 5$

$x_1, x_2 \geq 0$

⇒ The standard form of the given L.P.P. is

$\max z = 5x_1 + 8x_2 + 0 \cdot s_1 + 0 \cdot s_2 + 0 \cdot s_3 - MS_1 - MS_2$

s.t. $3x_1 + 2x_2 - s_1 + S_1 = 3$

$x_1 + 4x_2 - s_2 + S_2 = 4$

$x_1 + x_2 + s_3 = 5$

$x_1, x_2, s_1, s_2, s_3, S_1, S_2 \geq 0$

Where, s_1, s_2 are surplus variables, s_3 is slack variable, S_1, S_2 are artificial variables

M is a large +ve real number.

Table - I

		C_j	5	8	0	0	0	$-M$	$-M$	
C_B	x_B	b	x_1	x_2	s_1	s_2	s_3	S_1	S_2	
$-M$	S_1	3	3	2	-1	0	0	1	0	$\frac{3}{2}$
$-M$	S_2	4	1	$\boxed{4}$	0	-1	0	0	1	1*
0	s_3	5	1	1	0	0	1	0	0	5
$z_j - C_j$			$-4M-5$	$-6M-8$	M	M	0	0	0	
				↑					↓	

Table-2

C_B	x_B	C_j / b	5 x_1	8 x_2	0 s_1	0 s_2	0 s_3	$-M$ S_1	
$-M$	S_1	1	$\boxed{\frac{5}{2}}$	0	-1	$\frac{1}{2}$	0	1	$\frac{2}{5}$ ✓
8	x_2	1	$\frac{1}{4}$	1	0	$-\frac{1}{4}$	0	0	4
0	s_3	4	$\frac{3}{4}$	0	0	$\frac{1}{4}$	1	0	$\frac{16}{3}$
$z_j - C_j$			$-\frac{5}{2}M-3$	0	M	$-\frac{M}{2}-2$	0	0	
			↑					↓	

Table-3

C_B	x_B	C_j / b	5 x_1	8 x_2	0 s_1	0 s_2	0 s_3	
5	x_1	$\frac{2}{5}$	1	0	$-\frac{2}{5}$	$\boxed{\frac{1}{5}}$	0	2*
8	x_2	$\frac{9}{10}$	0	1	$\frac{1}{10}$	$-\frac{3}{10}$	0	
0	s_3	$\frac{37}{10}$	0	0	$\frac{3}{10}$	$\frac{1}{10}$	1	37
$z_j - C_j$			0	0	$-\frac{6}{5}$	$-\frac{7}{5}$	0	
			↓			↑		

Table-4

C_B	x_B	C_j / b	5 x_1	8 x_2	0 s_1	0 s_2	0 s_3	
0	s_2	2	5	0	-2	1	0	
8	x_2	$\frac{3}{2}$	$\frac{3}{2}$	1	$-\frac{1}{2}$	0	0	
0	s_3	$\frac{7}{2}$	$-\frac{1}{2}$	0	$\boxed{\frac{1}{2}}$	0	1	7*
$z_j - C_j$			7	0	-4	0	0	
					↑		↓	

Problem:-

1) Write down the dual of the following Primal:

$\max z = 2x_1 + 3x_2$

s.t. $x_1 - x_2 \le 3$

$2x_1 + 4x_2 \le 5$

$x_1 + x_2 \le 2$

$x_1, x_2 \ge 0$

⇒ The dual of the given primal is,

$\min \omega = 3v_1 + 5v_2 + 2v_3$

s.t. $v_1 + 2v_2 + v_3 \ge 2$

$-v_1 + 4v_2 + v_3 \ge 3$

$v_1, v_2, v_3 \ge 0$

2) Write down the dual of the following primal:

$\max z = 3x_1 - 2x_2 + x_3$

s.t. $x_1 + 5x_2 - x_3 \le 3$

$-2x_1 + x_2 - 4x_3 \ge 1$

$x_1, x_2, x_3 \ge 0$

⇒ The given primal can be written as,

$\max z = 3x_1 - 2x_2 + x_3$

s.t. $x_1 + 5x_2 - x_3 \le 3$

$2x_1 - x_2 + 4x_3 \le -1$

$x_1, x_2, x_3 \ge 0$

∴ The dual is,

$\min \omega = 3v_1 - v_2$

s.t. $v_1 + 2v_2 \ge 3$

$5v_1 - v_2 \ge -2$

$-v_1 + 4v_2 \ge 1$

$v_1, v_2 \ge 0$

3) Write down the dual of the following Primal:

$\min z = x_1 + 2x_2 + 3x_3 - x_4$

s.t. $3x_1 - x_2 + x_3 - x_4 \geq 5$

$2x_1 + x_2 - x_3 + 5x_4 \leq 7$

$x_1, x_2, x_3, x_4 \geq 0$

⇒ The given primal can be written as,

$\min z = x_1 + 2x_2 + 3x_3 - x_4$

s.t. $3x_1 - x_2 + x_3 - x_4 \geq 5$

$-2x_1 - x_2 + x_3 - 5x_4 \geq -7$

$x_1, x_2, x_3, x_4 \geq 0$

∴ The dual is,

$\max w = 5v_1 - 7v_2$

s.t. $3v_1 - 2v_2 \leq 1$

$-v_1 - v_2 \leq 2$

$v_1 + v_2 \leq 3$

$-v_1 - 5v_2 \leq -1$

$v_1, v_2 \geq 0$

4) Find the dual of the Primal problem:

$\min z = 5x_1 - 7x_2 + 3x_3$

s.t. $x_1 - 2x_2 - x_3 \leq 3$

$5x_1 - x_2 + x_3 \geq 4$

$x_1 + x_2 - x_3 = 7$

$x_1, x_2, x_3 \geq 0$

⇒ The given L.P.P. can be written as,

$\min z = 5x_1 - 7x_2 + 3x_3$

s.t. $-x_1 + 2x_2 + x_3 \geq -3$

$5x_1 - x_2 + x_3 \geq 4$

$x_1 + x_2 - x_3 = 7$

$x_1, x_2, x_3 \geq 0$

∴ The dual is,

$\max w = -3v_1 + 4v_2 + 7v_3$

s.t. $-v_1 + 5v_2 + v_3 \leq 5$

$2v_1 - v_2 + v_3 \leq -7$

$v_1 + v_2 - v_3 \leq 3$

$v_1, v_2 \geq 0$ and v_3 is unrestricted in sign.

5) write down the dual of the following Primal :

$\max z = x_1 + 2x_2 - 3x_3 + 5x_4$

s.t. $2x_1 - 3x_2 + x_3 - 2x_4 \leq 2$

$x_1 + x_2 + x_3 - x_4 = 3$

$3x_1 + 2x_2 - 5x_3 + 2x_4 \geq 2$

$x_1 - x_2 + x_3 - x_4 = 1$

$x_1, x_2, x_3, x_4 \geq 0$

=> The given primal can be writton as,

$\max z = x_1 + 2x_2 - 3x_3 + 5x_4$

s.t. $2x_1 - 3x_2 + x_3 - 2x_4 \leq 2$

$x_1 + x_2 + x_3 - x_4 = 3$

$-3x_1 - 2x_2 + 5x_3 - 2x_4 \leq -2$

$x_1 - x_2 + x_3 - x_4 = 1$

$x_1, x_2, x_3, x_4 \geq 0$

∴ The dual is,

$\min \omega = 2v_1 + 3v_2 - 2v_3 + v_4$

s.t. $2v_1 + v_2 - 3v_3 + v_4 \geq 1$

$-3v_1 + v_2 - 2v_3 - v_4 \geq 2$

$v_1 + v_2 + 5v_3 + v_4 \geq -3$

$-2v_1 - v_2 - 2v_3 - v_4 \geq 5$

$v_1, v_3 \geq 0$ and v_2 and v_4 are unrestricted in si

6) write down the dual of the following Primal !

$\min z = x_1 + 2x_2 - x_3$

s.t. $2x_1 - x_2 + x_3 \geq 3$

$5x_1 + x_2 - 7x_3 \leq 2$

$x_1, x_3 \geq 0$, x_2 is unrestricted in sign.

=> The given Primal can be written as,

$\min z = x_1 + 2x_2 - x_3$

s.t. $2x_1 - x_2 + x_3 \geq 3$

$-5x_1 - x_2 + 7x_3 \geq -2$

$x_1, x_3 \geq 0$, x_2 is unrestricted in sign.

∴ The dual is,

$\max \omega = 3v_1 - 2v_2$

s.t. $2v_1 - 5v_2 \leq 1$

$-v_1 - v_2 = 2$

$v_1 + 7v_2 \leq -1$

$v_1, v_2 \geq 0$

Problem :-

1) Use duality to find the optimal solution of the following L.P.P. $\min z = 15x_1 + 10x_2$

$$\left.\begin{aligned} \text{s.t.}\quad & 3x_1 + 5x_2 \geq 5 \\ & 5x_1 + 2x_2 \geq 3 \\ & x_1, x_2 \geq 0 \end{aligned}\right\} \text{——} (i)$$

⇒ The dual of the given primal is,

$$\left.\begin{aligned} & \max w = 5v_1 + 3v_2 \\ \text{s.t.}\quad & 3v_1 + 5v_2 \leq 15 \\ & 5v_1 + 2v_2 \leq 10 \\ & v_1, v_2 \geq 0 \end{aligned}\right\} \text{——} (ii)$$

The standard form of (ii) is,

$$\max w = 5v_1 + 3v_2 + 0.s_1 + 0.s_2$$

$$\text{s.t.}\quad 3v_1 + 5v_2 + s_1 = 15$$

$$5v_1 + 2v_2 + s_2 = 10$$

$$v_1, v_2, s_1, s_2 \geq 0$$

Where s_1 and s_2 are slack variables.

Table-1

		c_j	5	3	0	0	
C_B	X_B	b	v_1	v_2	s_1	s_2	
0	s_1	15	3	5	1	0	5
0	s_2	10	$\boxed{5}$	2	0	1	2*
$w_j - c_j$			−5	−3	0	0	
			↑			↓	

Table 2

		c_j	5	3	0	0	
C_B	X_B	b	v_1	v_2	s_1	s_2	
0	s_1	9	0	$\boxed{\frac{19}{5}}$	1	$-3/5$	$\frac{45}{19}$*
5	v_1	2	1	$2/5$	0	$1/5$	5
$w_j - c_j$			0	−1	0	1	
				↑	↓		

$$R_1' = R_1 - \frac{3}{5}R_2$$

$$5 - \frac{6}{5} = \frac{19}{5}$$

Table-3

		c_j	5	3	0	0
C_B	X_B	b	v_1	v_2	s_1	s_2
3	v_2	$\frac{45}{19}$	0	1	$\frac{5}{19}$	$\frac{-3}{19}$
5	v_1	$\frac{20}{19}$	1	0	$-\frac{2}{19}$	$\frac{5}{19}$
$\omega_j - a_j$			0	0	$\frac{5}{19}$	$\frac{16}{19}$

$2 - \frac{13}{19} = \frac{25}{19}$

$\frac{1}{5} + \frac{2}{19} \cdot \frac{9}{5}$

$\frac{1}{5}\left(1 + \frac{6}{19}\right)$

$\frac{1}{5} \times \frac{25}{19} = \frac{5}{19}$

$\frac{13}{19} - \frac{10}{19}$

$-\frac{9}{19} \quad \frac{125}{19}$

Since, all net evaluations are +ve, table-3 is the optimal table.

The optimal solution of the dual problem (ii) is $v_1 = \frac{20}{19}$, $v_2 = \frac{45}{19}$

$\therefore \omega_{max} = \frac{235}{19}$

~~$\therefore x_1 = \frac{5}{19}, \quad x_2 = \frac{16}{19}$~~

~~$\therefore z_{min} =$~~

(iii) Optimal solution of primal problem :—

s_1 and s_2 are the initial basic variables.

$\therefore$ The net-evaluations in the optimal table-3 corresponding to s_1 and s_2 are the values of the original ~~va~~ variable respectively.

$\therefore x_1 = \frac{5}{19}, \quad x_2 = \frac{16}{19}, \quad z_{min} = \frac{235}{19}$

2) ~~Let us~~

[Note:— Let us now solve the primal problem (i).

$\max z^* (= -z) = -15x_1 - 10x_2 + 0.s_1 + 0.s_2 - MS_1 - MS_2$

s.t. $3x_1 + 5x_2 - s_1 + S_1 = 5$

$5x_1 + 2x_2 - s_2 + S_2 = 3$

$x_1, x_2, s_1, s_2, S_1, S_2 \geq 0$

where, s_1, s_2 are surplus variables and S_1, S_2 are artificial variables.

Table-1

C_B	X_B	c_j / b	-15 x_1	-10 x_2	0 s_1	0 s_2	-M S_1	-M S_2	
-M	S_1	5	3	5	-1	0	1	0	5/3
-M	S_2	3	$\boxed{5}$	2	0	-1	0	1	$\frac{3}{5}$ *
$z_j - c_j$			-8M +15	-7M +10	M	M	0	0	
			↑					↓	

Table-2

C_B	X_B	c_j / b	-15 x_1	-10 x_2	0 s_1	0 s_2	-M S_1	-M S_2
-M	S_1	$\frac{16}{5}$	0	$\boxed{\frac{19}{5}}$	-1	$\frac{3}{5}$	1	$-\frac{3}{5}$
-15	x_1	$\frac{3}{5}$	1	$2/5$	0	$-1/5$	0	$\frac{1}{5}$
$z_j - c_j$			0	$-\frac{19}{5}M+4$	M	$-\frac{3}{5}M+3$	0	$\frac{8}{5}M-3$
				↑			↓	

Table-3

C_B	X_B	c_j / b	-15 x_1	-10 x_2	0 s_1	0 s_2	-M S_1	-M S_2
-10	x_2	$\frac{16}{19}$	0	1	$-\frac{5}{19}$	$\frac{3}{19}$	$\frac{5}{19}$	$-\frac{3}{19}$
-15	x_1	$\frac{5}{19}$	1	0	$\frac{2}{19}$	$-\frac{5}{19}$	$-\frac{2}{19}$	$\frac{5}{19}$
$z_j - c_j$			0	0	$\frac{20}{19}$	$\frac{45}{19}$	$M-\frac{20}{19}$	$M-\frac{45}{19}$

$R_1' = R_1 - \frac{3}{5} R_2$

$5 - \frac{9}{5}$

$5 - \frac{6}{5} = \frac{19}{5}$

$-6+10$

$\frac{3}{5} - 1$

$\frac{3}{5} - 1$

$R_2' = R_2 - \frac{2}{19} R_1$

$\frac{3}{5} - \frac{16}{5} \cdot \frac{2}{19}$

$\frac{50}{19} - \frac{30}{19}$ $\quad \frac{1}{5}\left(3 - \frac{32}{19}\right)$

$\frac{20}{19}$ $\quad \frac{1}{5} \times \frac{57-32}{19}$

$-\frac{30}{19} + \frac{75}{19}$

$= \frac{45}{19}$ $\quad \frac{1}{5} \times \frac{25}{19} = \frac{5}{19}$

$-\frac{50}{19} + \frac{30}{19}$ $\quad -\frac{1}{5} - \frac{2}{19} \cdot \frac{3}{5}$

$\frac{30}{19} - \frac{75}{19}$ $\quad = -\frac{1}{5}\left(1 + \frac{6}{19}\right)$

$-\frac{45}{19}$ $\quad = -\frac{1}{5} \times \frac{25}{19}$

$\frac{1}{5} + \frac{2}{19} \cdot \frac{3}{5}$

$\frac{1}{5}\left(1 + \frac{6}{19}\right)$

$= \frac{1}{5} \times \frac{25}{19}$

$\because$ all $z_j - c_j \geq 0$, table-3 is the optimal table of the primal problem (i).

$\therefore$ optimal solution of primal problem (i),

$x_1 = \frac{5}{19}$, $x_2 = \frac{16}{19}$ and $z_{min} = \frac{235}{19}$

(iii) solution of dual problem (ii) :-

In optimal table-3, the net evaluations of the initial basic variables are respectively $M - \frac{20}{19}$ and $M - \frac{45}{19}$

$\therefore v_1 = \frac{20}{19}$, $v_2 = \frac{45}{19}$ [Putting M=0 and multiplying by -1)

Transportation Problem (T.P.)

Problem :-

1) Find the initial basic feasible solution of the following transportation problem :-

	D_1	D_2	D_3	D_4	a_i
O_1	19	20	50	10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
b_j	5	8	7	14	34 34

1) North-West corner method :-

Table - 1

	D_1	D_2	D_3	D_4	a_i
O_1	$\boxed{5}$ 19	20	50	10	$\cancel{7}$ 2
O_2	70	30	40	60	9
O_3	40	8	70	20	18
b_j	5	8	7	14	34 34

table - 2

	D_2	D_3	D_4	
O_1	$\boxed{2}$ 20	50	10	2
O_2	30	40	60	9
O_3	8	70	20	18
	$\cancel{8}$ 6	7	14	

table - 3

	D_2	D_3	D_4	
O_2	$\boxed{6}$ 30	40	60	$\cancel{9}$ 3
O_3	8	70	20	18
	6	7	14	

table - 4

	D_3	D_4	
O_2	$\boxed{3}$ 40	60	3
O_3	$\boxed{4}$ 70	$\boxed{14}$ 20	18
	$\cancel{7}$ 4	14	

∴ Initial B.F.S. is $x_{11} = 5$, $x_{12} = 2$, $x_{22} = 6$, $x_{23} = 3$, $x_{33} = 4$, $x_{34} = 14$

∴ corresponding cost is,

$$5 \times 19 + 2 \times 20 + 6 \times 30 + 3 \times 40 + 4 \times 70 + 14 \times 20$$

$$= 95 + 40 + 180 + 120 + 280 + 280$$

$$= 995$$

2) Matrix - minima method (least cost entry method) :-

Table - 1

	D_1	D_2	D_3	D_4	a_i
O_1	19	20	50	10	7
O_2	70	30	40	60	9
O_3	40	$\boxed{8}$ 8	70	20	~~18~~ 10
b_j	5	8	7	14	34 34

table - 2

	D_1	D_3	D_4	
O_1	19	50	$\boxed{7}$ 10	7
O_2	70	40	60	9
O_3	40	70	20	10
	5	7	~~14~~ 7	

table - 3

	D_1	D_3	D_4	
O_2	70	40	60	9
O_3	40	70	$\boxed{7}$ 20	~~10~~ 3
	5	7	7	

table - 4

	D_1	D_3	
O_2	$\boxed{2}$ 70	$\boxed{7}$ 40	~~9~~ 7
O_3	$\boxed{3}$ 40	70	3
	~~5~~ 2	7	

∴ Initial basic feasible solution is,

$x_{32} = 8,\ x_{14} = 7,\ x_{34} = 7,\ x_{31} = 3,$

$x_{21} = 2,\ x_{23} = 7$

∴ Corresponding cost $= 8\times 8 + 7\times 10 + 7\times 20 + 2\times 70 + 7\times 40 + 3\times 40$

$= 64 + 70 + 140 + 140 + 280 + 120$

$= 814$

3) Row - minima method :-

Table - 1

	D_1	D_2	D_3	D_4	a_i
O_1	19	20	50	$\boxed{7}$ 10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
b_j	5	8	7	~~14~~ 7	

Table - 2

	D_1	D_2	D_3	D_4	
O_2	70	$\boxed{8}$ 30	40	60	~~9~~ 1
O_3	40	8	70	20	18
	5	8	7	7	

[illegible]

Table-3

	D_1	D_3	D_4	
O_2	70	$\boxed{1}$ 40	60	1
O_3	$\boxed{5}$ 40	$\boxed{6}$ 70	$\boxed{7}$ 20	18
	5	~~7~~ 6	7	

∴ Initial basic feasible solution is, $x_{14} = 7$, $x_{22} = 8$, $x_{23} = 1$, $x_{31} = 5$, $x_{33} = 6$, $x_{34} = 7$

∴ Corresponding cost $= 70 + 240 + 40 + 200 + 420 + 140$

$= 1110$

4) Column-minima method :-

Table-1

	D_1	D_2	D_3	D_4	a_i
O_1	$\boxed{5}$ 19	20	50	10	~~7~~ 2
O_2	70	30	40	60	9
O_3	40	8	70	20	18
b_j	5	8	7	14	

Table-2

	D_2	D_3	D_4	
O_1	20	50	10	2
O_2	30	40	60	9
O_3	$\boxed{8}$ 8	70	20	~~18~~ 10
	8	7	14	

Table-3

	D_3	D_4	
O_1	50	$\boxed{2}$ 10	2
O_2	$\boxed{7}$ 40	$\boxed{2}$ 60	~~9~~ 2
O_3	70	$\boxed{10}$ 20	10
	7	14	

∴ Initial basic feasible solution is, $x_{11} = 5$, $x_{32} = 8$, $x_{23} = 7$, $x_{14} = 2$, $x_{24} = 2$, $x_{34} = 10$

∴ Corresponding cost is $= 95 + 64 + 280 + 20 + 120 + 200$

$= 779$

5) Vogel's Approximation Method (VAM) (Unit Penalty method) :-

Table - 1

	D_1	D_2	D_3	D_4	a_i
O_1	$\boxed{5}$ 19	20	50	10	~~7~~ 2 (9)
O_2	70	30	40	60	9 (10)
O_3	40	8	70	20	18 (12)
b_j	5 (21)	8 (12)	7 (10)	14 (10)	

Table - 2

	D_2	D_3	D_4	
O_1	20	50	10	2 (10)
O_2	30	40	60	9 (10)
O_3	$\boxed{8}$ 8	70	20	~~18~~ 10 (12)
	8 (12)	7 (10)	14 (10)	

Table - 3

	D_3	D_4	
O_1	50	10	2 (40)
O_2	40	60	9 (20)
O_3	70	$\boxed{10}$ 20	10 (50)
	7 (10)	~~14~~ 4 (10)	

Table - 4

	D_3	D_4	
O_1	50	$\boxed{2}$ 10	2 (40)
O_2	$\boxed{7}$ 40	$\boxed{2}$ 60	9 (20)
	7 (10)	~~4~~ 2 (50)	

∴ Initial B.F.S. is,

$x_{11} = 5$, $x_{32} = 8$, $x_{34} = 10$,

$x_{14} = 2$, $x_{23} = 7$, $x_{24} = 2$

∴ Corresponding cost is $= 95 + 64 + 200 + 20 + 280 + 120$

$= 779$

For the 1st row the difference among minimum and 2nd minimum is 9 which is known as Penalty. The penalties of each row and column are calculated and are shown in the parenthesis.

Let us now choose the maximum penalty. Here the maximum penalty is 21 which corresponds to the 1st column.

∴ we allocate in the minimum cost cell of the 1st column. The minimum cost cell is C_{11}.

∴ we allocate min(7, 5) in this cell. Due to this allocation the demand of D_1 is fulfilled and the availability of O_1 is decreased to $2 (= 7 - 5)$. Since, the demand of D_1 is fulfilled, we omit the 1st column in table-2. Repeating this process, we get table-2, table-3, table-4 and the initial B.F.S.

Q) Determine the initial B.F.S. of the transportation problem:-

1)

From \ To	D_1	D_2	D_3	
O_1	2	7	4	5
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
	7	9	18	34

2)

	D_1	D_2	D_3	
O_1	5	1	7	10
O_2	6	4	6	80
O_3	3	2	5	15
	75	20	50	105

145

1) i) North-West corner method 34

Table-1

	D_1	D_2	D_3	
O_1	$\boxed{5}$ 2	7	4	5
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
	~~7~~ 2	9	18	

Table-2

	D_1	D_2	D_3	
O_2	$\boxed{2}$ 3	3	1	~~8~~ 6
O_3	5	4	7	7
O_4	1	6	2	14
	2	9	18	

Table-3

	D_2	D_3	
O_2	$\boxed{6}$ 3	1	6
O_3	4	7	7
O_4	6	2	14
	~~9~~ 3	18	

Table-4

	D_2	D_3	
O_3	$\boxed{3}$ 4	$\boxed{4}$ 7	~~7~~ 4
O_4	6	$\boxed{14}$ 2	14
	3	13	

∴ Initial B.F.S. is, $x_{11} = 5$, $x_{21} = 2$, $x_{22} = 6$, $x_{32} = 3$, $x_{33} = 4$, $x_{43} = 14$

∴ Corresponding cost $= 10 + 6 + 18 + 12 + 28 + 28 = 102$

ii) Matrix minima method:-

Table-1

	D_1	D_2	D_3	
O_1	2	7	4	5
O_2	3	3	$\boxed{8}$ 1	8
O_3	5	4	7	7
O_4	1	6	2	14
	7	9	~~18~~ 10	

Table-2

	D_1	D_2	D_3	
O_1	2	7	4	5
O_3	5	4	7	7
O_4	$\boxed{7}$ 1	6	2	~~14~~ 7
	7	9	10	

Table-3

	D_2	D_3	
O_1	7	4	5
O_3	4	7	7
O_4	6	$\boxed{7}$ 2	7
	9	~~10~~ 3	

Table-4

	D_2	D_3	
O_1	$\boxed{2}$ 7	$\boxed{3}$ 4	~~5~~ 2
O_3	$\boxed{7}$ 4	7	7
	9	3	

∴ Initial B.F.S. is, $x_{23} = 8$, $x_{41} = 7$, $x_{43} = 7$, $x_{12} = 2$, $x_{13} = 3$, $x_{32} = 7$

∴ Corresponding cost $= 8 + 7 + 14 + 14 + 12 + 28$
$= 83$

Optimality test

Problem :-

1) Solve the following transportation problem :

Destination

Origin	D_1	D_2	D_3	D_4	a_i
O_1	19	14	23	11	11
O_2	15	16	12	21	13
O_3	30	25	16	39	18
b_j	6	10	11	15	42 42

⟹

	D_1	D_2	D_3	D_4	a_i			
O_1	19	14	23	$\boxed{11}$ 11	11 (3)			
O_2	$\boxed{6}$ 15	$\boxed{3}$ 16	12	$\boxed{4}$ 21	13 (3)	9 ~~13~~(3)	3 ~~9~~ (3)	3(4)
O_3	30	$\boxed{7}$ 25	$\boxed{11}$ 16	39	18 (9)	18 (9)	18 (9)	18 (9)
b_j	6	10	11	~~15~~ 4				
	(9)	(2)	(4)	(10)				
	6	10	11	4				
	(15)	(9)	(4)	(18)				
	6	10	11					
	(15)	(9)	(4)					
		~~10~~ 7	11					
		(9)	(4)					

Let us use VAM method to get initial B.F.S.

The initial B.F.S. is, $x_{14} = 11$, $x_{21} = 6$, $x_{22} = 3$, $x_{24} = 4$, $x_{32} = 7$, $x_{33} = 11$

~~∴ Corresponding cost = 121 + 90 + 48 + 84 +~~

(III) Optimality test :-

Table-I

	D_1	D_2	D_3	D_4	a_i	u_i
O_1	⑭ 19	⑧ 14	㉖ 23	$\boxed{11}$ 11	11	-10
O_2	$\boxed{6}$ 15	$\boxed{3}$ 16	⑤ 12	$\boxed{4}$ 21	13	⓪ 0
O_3	⑥ 30	$\boxed{7}$ 25	$\boxed{11}$ 16	⑨ 39	18	9
b_j	6	10	11	15	42 42	
v_j	15	16	7	21		

Since, $m+n-1 = 6 =$ no. of allocations

∴ The transportation problem is non-degenerate.

Let us now evaluate u_i and v_j.

Since, in 2nd row, there are maximum no. of allocations, we take $u_2 = 0$. Let us now use the formula $c_{ij} = u_i + v_j$ for the occupied cells to find the values of the other variables. The values of the variables are shown in the table – 1.

Let us now calculate the cells evaluations Δ_{ij} for the unoccupied cells using the formula $\Delta_{ij} = c_{ij} - (u_i + v_j)$, which are shown in the table within circle.

Since, all $\Delta_{ij} \geq 0$, the initial B.F.S. is the optimal solution.

∴ The optimal solution is, $x_{14} = 11$, $x_{21} = 6$, $x_{22} = 3$, $x_{24} = 4$, $x_{32} = 7$, $x_{33} = 11$

and the optimal cost is $= 121 + 90 + 48 + 84 + 175 + 176$

$= 694$

2) Solve the following transportation problem :–

	D_1	D_2	D_3	D_4	a_i
O_1	19	20	50	10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
b_j	~~8~~5	8	7	14	34 34

⇒ Table-1

	D_1	D_2	D_3	D_4	a_i			
O_1	$\boxed{5}$ 19	20	50	$\boxed{2}$ 10	~~7~~ 2 (9)	2 (10)	2 (40)	2 (40)
O_2	70	30	$\boxed{7}$ 40	$\boxed{2}$ 60	9 (10)	9 (10)	9 (20)	9 (20)
O_3	40	$\boxed{8}$ 8	70	$\boxed{10}$ 20	18 (12)	~~18~~ 10 (12)	10 (50)	
b_j	5	8	7	14	34 34			
	(21)	(12)	(10)	(10)				

	D_2	D_3	D_4
	8 (12)	7 (10)	14 (10)
		7 (10)	~~14~~ 4 (10)
		7 (10)	~~4~~ 2 (50)

Using VAM method the initial B.F.S. is,

$x_{11}=5$, $x_{14}=2$, $x_{23}=7$, $x_{24}=2$, $x_{32}=8$, $x_{34}=10$

Ⅲ **Optimality test :-**

Table - 1

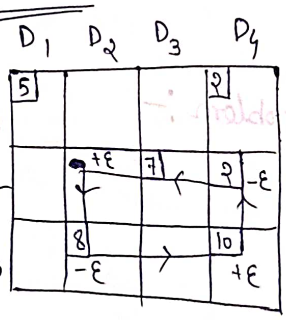
	D_1	D_2	D_3	D_4	a_i	u_i
O_1	$\boxed{5}$ 19	(22) 20	(60) 50	$\boxed{2}$ 10	7	10
O_2	① 70	(-18) 30	$\boxed{7}$ 40	$\boxed{2}$ 60	9	60
O_3	⑪ 40	$\boxed{8}$ 8	(70) 70	$\boxed{10}$ 20	18	20
b_j	5	8	7	14	34 \ 34	
v_j	9	-12	-20	0		

Since, $m+n-1=6=$ no. of allocations.

∴ The given T.P. is non-degenerate.

Since, $\Delta_{22}=-18<0$, the initial B.F.S. is not optimal.

Table-2



	D_1	D_2	D_3	D_4
O_1	$\boxed{5}$			$\boxed{2}$
O_2		$+\varepsilon$	$\boxed{7}$	$\boxed{2}$ $-\varepsilon$
O_3		$\boxed{8}$ $-\varepsilon$		$\boxed{10}$ $+\varepsilon$

Table - 3

	D_1	D_2	D_3	D_4	a_i	u_i
O_1	$\boxed{5}$ 19	(22) 20	(42) 50	$\boxed{2}$ 10	7	0
O_2	⑲ 70	$\boxed{2}$ 30	$\boxed{7}$ 40	⑱ 60	9	32
O_3	⑪ 40	$\boxed{6}$ 8	(52) 70	$\boxed{12}$ 20	18	10
b_j	5	8	7	14	34 \ 34	
v_j	19	-2	8	10		

Since, all $\Delta_{ij} \geq 0$, table-3 is the optimal table.

∴ The optimal solution is, $x_{11}=5$, $x_{14}=2$,
$x_{22}=2$, $x_{23}=7$, $x_{32}=6$, $x_{34}=12$

∴ The optimal cost is $=95+20+60+280+48+240$
$=743$

[**Note**:- The method is known as MODI method (MODified DIstribution).]

Travelling Salesman Problem

Problem:-

1) Solve the following travelling Salesman Problem :-

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & \infty & 6 & 12 & 6 & 4 \\ 2 & 6 & \infty & 10 & 5 & 4 \\ 3 & 8 & 7 & \infty & 11 & 3 \\ 4 & 5 & 4 & 11 & \infty & 5 \\ 5 & 5 & 2 & 7 & 8 & \infty \end{array}$$

X

⇒

Route	Cost
1→5→2→4→3→1	4+2+5 11+8=30
2→5→1→4→3→2	4+5+6+11+7 = 33
3→5→2→4→1→3	3+2+5+5 +12 = 27 *
4→2→5→1→3→4	4+4+5+ 12+11=36
5→2→4→1→3→5	2+4+5 +12+3 = 27 *

∴ Optimal route is 3→5→2→4→1→3 and
5→2→4→1→3→5

2) Solve the following travelling Salesman problem :-

$$\begin{array}{cc|cccccc} & & & & \text{To} & & & \\ & & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline & 1 & \infty & 5 & 13 & 6 & 4 & 8 \\ & 2 & 6 & \infty & 11 & 5 & 4 & 3 \\ & 3 & 8 & 7 & \infty & 6 & 3 & 11 \\ \text{From} & 4 & 5 & 4 & 11 & \infty & 5 & 8 \\ & 5 & 5 & 2 & 7 & 8 & \infty & 4 \\ & 6 & 6 & 3 & 13 & 5 & 4 & \infty \end{array}$$

X

⇒ ~~Route~~ ~~Cost~~

Starting city	Route	Cost
1	1→5→2→6→4→3 1 ←	4+2+3+5+11 +8 = 33
2	2→6→5→1→4 2 ← 3 ←	3+4+5+6 +11+7 = 36
3	3→5→2→6→4 3 ← 1	3+2+3+5 * +5+13=31
4	4→2→6→5→1 4 ← 3 ←	4+3+4+5+ 13+6 =35
5	5→2→6→4 5 ← 3 ← 1 ←	2+3+5+5 +13+3 * = 31
6	6→2→5→1 6 ← 3 ← 4 ←	3+4+5+ 6+11+11 = 40

optimal route is 3→5→2→6→4→1→3

3) Solve the following travelling Salesman problem:–

$$\begin{array}{c|cccc} & A & B & C & D \\ \hline A & \infty & 4 & 7 & 3 \\ B & 4 & \infty & 6 & 3 \\ C & 7 & 6 & \infty & 7 \\ D & 3 & 3 & 7 & \infty \end{array}$$

⇒

Starting city	Route	Cost
A	A → D → B → C → A	3+3+6+7 = 19*
B	B → D → A → C → B	3+3+7+6 = 19*
C	C → B → D → A → C	6+3+3+7 = 19*
D	D → A → B → C → D D → B → A → C → D	3+4+6+7 = 20 3+4+7+7 = 21

∴ optimal route are A → D → B → C → A and B → D → A → C →

4) Solve the following ~~~~ travelling salesman problem:–

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & - & 7 & 6 & 8 & 4 \\ B & 7 & - & 8 & 5 & 6 \\ C & 6 & 8 & - & 9 & 7 \\ D & 8 & 5 & 9 & - & 8 \\ E & 4 & 6 & 7 & 8 & - \end{array}$$

(min cost = 30)

5)

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & \infty & 4 & 7 & 3 & 4 \\ B & 4 & \infty & 6 & 3 & 4 \\ C & 7 & 6 & \infty & 7 & 5 \\ D & 3 & 3 & 7 & \infty & 7 \\ E & 4 & 4 & 5 & 7 & \infty \end{array}$$

(min cost = 21)

4)

Starting city	Route	Cost
A	A → E → B → D → C → A	4+6+5+9+6 = 30*
B	B → D → A → E → C → B B → D → E → A → C → B	5+8+4+7+8 = 32 5+8+4+6+8 = 31
C	C → A → E → B → D → C	6+4+6+5+9 = 30*
D	D → B → E → A → C → D	5+6+4+6+9 = 30*
E	E → A → C → B → D → E	4+6+8+5+8 = 31

∴ Optimal route are, A → E → B → D → C → A and D → B → E → A → C → D