Integers

Well ordering principle :- Let, S be non empty subset of the set of natural number N then S has a smallest member.

Let, S be a non empty subset of N then x is smallest / least member of S if i) $x \in S$, ii) $x \leq a \ \forall a \in S$

if x be smallest member of S then $x-1 \notin S$

(I) **Principle of induction :-** Let, S be a subset of N such that i) $1 \in S$ ii) $k \in S \Rightarrow k+1 \in S$ then $S = N$

Let, $t = N - S$, we shall show that t is empty that is $t = \phi$

Let, if possible $t \neq \phi$

clearly, t is a nonempty subset of N. Therefore, by well ordering principle, t has smallest member m (say). Since, $1 \in S$ therefore $m > 1$ and hence $m-1$ is a positive integer.

Since, m is the smallest member of t, $m-1 \notin t$, therefore $m-1 \in S$

$\therefore$ by condition (ii) $m-1+1 \in S \Rightarrow m \in S$

which is a contradiction. Therefore, $t = \phi$

(II) Let, E_1 a statement involving a natural number N if i) E_1 is true and ii) E_k is true $\Rightarrow E_{k+1}$ is true. Then the statement E_N is true for all natural numbers N.

(II) Let, S be the set of natural number for which the E_n is true.

Therefore, S has the following property

i) $K \in S$, ii) $K \in S \Rightarrow K+1 \in S$

Therefore, by the principle of induction $S = N$. Therefore the statement E_N is true for all natural no.

1) Show by the principle of mathematical induction $1+2+3+\dots+n = \frac{n(n+1)}{2}$ ... (i)

$\Rightarrow$ we have, $1 = \frac{1(1+1)}{2} = 1$

$\therefore$ The result is true for $n = 1$

Let, the result be true for $n = K$

$1+2+3+\dots+K+(K+1) = \frac{K(K+1)}{2} + (K+1)$ [by (i)]

$= \frac{\cancel{(K+1)(K+1)}}{2} = \frac{(1+K)(K+1+1)}{2}$

$\therefore$ The result is true for $n = k+1$. Therefore by principle of induction $1+2+3+\dots+n = \frac{n(n+1)}{2}$ $\forall n \in N$

2) Prove that $3^{2n} - 8n - 1$ is divisible by 64 $\forall$ N.

$\rightarrow$ Let, $f(n) = 3^{2n} - 8n - 1$

we have, $f(1) = 9 - 8 - 1 = 0$

$\therefore f(1)$ is divisible by 64.

Let, $f(k)$ be divisible by 64

Now we have, $f(k+1) - f(k) = 3^{2k+2} - 8(k+1) - 1 - (3^{2k} - 8k - 1)$

$= 9 \cdot 3^{2k} - 8 - 3^{2k} = 8 \cdot 3^{2k} - 8$

$= 8(9^k - 1)$

$= 8(9-1)(1+9+9^2+\dots+9^{k})$

$= 64p$; $p = 1+9+9^2+\dots+9^k$ is an integer.

$\therefore f(k+1)$ is divisible by 64.

$\therefore$ By the principle of induction $f(n)$ is divisible by 64, $n \in N$

Division Algorithm :-

For given integers a and b with $b > 0$, there exists unique integers q and r ~~since~~ such that $a = qb + r$, where $0 \le r < b$

Let, $S = \{a - bx, : x \in Z ; a - bx \geq 0$

Let us 1st show that S is non empty.

We have, $b \geq 1$

$b|a| \geq |a|$

$\Rightarrow a + b|a| \geq a + |a| \geq 0$

$\Rightarrow a - b(-|a|) \geq 0$

$\therefore a - b(-|a|) \in S$

$\therefore S$ is non empty

Let, r be the smallest member of S. Therefore we have.

$a - bq = r$, $r \geq 0$ and $q \in Z$

We shall show that $r < b$, Let it

Let, if Possible $r \geq b$

$\therefore$ we have, $a-(q+1)b = a-bq-b = r-b \geq 0$

$\therefore a-(q+1)b \in S$

again we have, $a-(q+1)b$

$= a-bq-b = r-b < r$ which is contradiction.

$\therefore r < b$

$\therefore a = bq + r, \quad 0 \leq r < b$

$r-b < r, \; h \neq b$

Finally we shall show that q and r are unique.

Let, $a = bq + r, \quad 0 \leq r < b$ — (i)

$a = bq_1 + r_1, \quad 0 \leq r_1 < b$ — (ii)

$\therefore$ from (i) and (ii) we get, $0 = b(q_1 - q) + (r_1 - r)$

$b(q-q_1) = r_1 - r$

$\Rightarrow |b|\,|q-q_1| = |r_1 - r|$

$\Rightarrow b(q-q_1) = |r_1 - r|$ — (iii)

from (i) and (ii) we have,

$$0 \leq r_1 < b.$$
$$-b < -r \leq 0$$
$$-b < r_1 - r < b$$
$$|r_1 - r| < b$$

$\therefore$ from (iii), $b|q-q_1| < b$

$(q-q_1) < 1$ — (iv)

Since, q and q_1 are both integers satisfying (iv).

$\therefore q = q_1$

$\therefore$ from (iii), $r = r_1$

Thus the uniqueness is true

q is called the quiotent and r is called the remainder in the division of $\frac{a}{b}$.

divisibility:— An integer a is said to be divisible by an integer $b (\neq 0)$ if there exists an integer c. such that $a = cb$. In this case, we say that b divides a and symbolically denoted by $b|a$.

Some properties:-

1) a/b and $b/c \Rightarrow a/c$

2) a/b and $b/a \Rightarrow a = \pm b$

(1) **Theorem:-** If a/b and a/c then $a/(bx+cy)$ for arbitary integers x and y.

$\Rightarrow$ Since a/b

$\therefore$ $b = ma$ for some $m \in \mathbb{Z}$

again since, a/c

$\therefore$ $c = na$ for some $n \in \mathbb{Z}$

$\therefore$ for any integers x and y we have,

$$bx+cy = max + nay = a(mx+ny)$$
$$= pa, \; p = mx+ny \in \mathbb{Z}.$$

$\therefore$ $a/(bx+cy)$

1) Prove that the square of an odd integer is of the form $8k+1$, where k is an integer.

$\Rightarrow$ By division algorithm, the remainder will be 0, 1, 2 or 3 upon division by 4,

$\therefore$ any integer can be expressed in any ~~of~~ one of the following forms.

$4q, \; 4q+1, \; 4q+2, \; 4q+3$, where q is an integer.

Among these forms $4q+1$ and $4q+3$ are odd.

$\therefore$ we have,

$$(4q+1)^2 = 16q^2 + 8q + 1 = 8(2q^2+q)+1 = 8k+1, \; k = 2q^2+q \in \mathbb{Z}$$

$$(4q+3)^2 = 16q^2+24q+9 = 16q^2+24q+8+1 = 8(2q^2+3q+1)+1$$
$$= 8k+1, \; k = 2q^2+3q+1 \in \mathbb{Z}$$

Common divisors

An integer d is said to be common divisor of the integers a and b if d/a and d/b

For example :- The divisors of 8 are 1, 2, 4 and 8

The divisors of 12 are 1, 2, 3, 4, 6 and 12.

∴ Common divisors of 8 and 12 are 1, 2 and 4.

(ii) 1 is a common divisor of any integers of a and b.

Greatest Common divisors :-

Let a and b be two integers not both zero. An integer d said to be greatest common divisor of a and b denoted by G.C.D. of a, b = d if

i) d/a and d/b $\Rightarrow c/d$

ii) c/a and c/b

For example, gcd of $(8, 12) = 4$

(iii) Let, a and b be two integers not both zero and $d = gcd(a, b)$ then ∃ integers x and y such that, $d = ax + by$.

Prime to each other (Relatively prime)

Diff. Two integers a and b not both zero are said to be prime to each other if $gcd(a, b) = 1$

For example The integers 8 and 15 are said to be prime to each other if $gcd(8, 15) = 1$

Theorem :- Two integers a and b not both zero are prime to each other if and only if there exist some integer x and y such that $ax + by = 1$

Proof :- Let us first suppose that a and b are prime to each other

∴ $gcd(a, b) = 1$ s.t. $1 = d = g(a, b) = ax + by$

∴ there exists 2 integer x and y

Conversely, suppose that there exists two integer x and y s.t. $ax + by = 1$

Let, $\gcd(a,b) = d$

$\therefore$ There exists integers u and v s.t. $d = au + bv$ for $u = x$ and $v = y$

$\therefore d = ax + by$
$= 1$

$\therefore$ a and b are prime to each other.

(ii) **Theorem:-** Let, $\gcd(a,b) = d$ (a and b are not both zero) then $\frac{a}{d}$ and $\frac{b}{d}$ are integers prime to each other.

Proof:- Since, $\gcd(a,b) = d$

$\therefore d/a$ and d/b

$\therefore$ there exists integer m and n s.t. $a = md$ and $b = nd$

$\therefore \frac{a}{d}$ and $\frac{b}{d}$ are both integers.

$\because \gcd(a,b) = d$

$\therefore$ There exists integer x and y s.t. $d = ax + by$

$1 = \frac{a}{d}x + \frac{b}{d}y$

$= mx + ny$

$\therefore \gcd(m,n) = 1$

$\therefore \frac{a}{d}$ and $\frac{b}{d}$ are relatively prime.

(iii) By division alternative, we have,

$72v + 30u = 12$

$72 = 30 \cdot 2 + 12$
$30 = 12 \cdot 2 + 6$
$12 = 6 \cdot 2 + 0$

$\therefore 12 = 6 \cdot 2 = (30 - 12 \cdot 2) \cdot 2 = 30 \cdot 2 - 12 \cdot 4$

$= 30 \cdot 2 - (72 - 30 \cdot 2)4$

$= 30 \cdot 2 - 72 \cdot 4 + 30 \cdot 8$

$= 30(10) + 72(-4)$

$\therefore u = 10$ and $v = -4$

1) Find two integers u and v satisfying $54u + 24v = 30$

$\Rightarrow$ By division alternative, we have,

$54 = 24 \cdot 2 + 6$
$24 = 6 \cdot 4 + 0$

$$\begin{array}{r|l} 24 & 54 \\ & 48 \quad (2 \\ \hline 6 & 24 \\ & 24 \quad (4 \\ \hline & 0 \end{array}$$

$\therefore 24 = 6 \cdot 4 = (54 - 2 \cdot 24 \cdot 2)4 = 54 \cdot 4 - 24 \cdot 8$
$= 54 \cdot 4 + 24(-8)$

$\therefore u = 4$ and $v = -8$

Prime integer

Diff An integer $p > 1$ is said to be prime integer if the only divisors of p are 1 and p.

Fundamental theorem of Arithmetic

Statement:- Any +ve integer is either 1 or prime or can be expressed as product of primes.

Euclid theorem:-

Statement:- The number of primes is infinite

Note:- Every +ve integers which are not prime can be expressed as $p = p_1^{\alpha_1}, p_2^{\alpha_2} \ldots p_m^{\alpha_m}$ where $p_1 < p_2 < \ldots < p_m$ are primes and $\alpha_1, \alpha_2 \ldots \alpha_m$ are positive integers.

Use principle of mathematical induction to prove that

1) $3^{2n-1} + 2^{n+1}$ is divisible by 7.

$\Rightarrow$ Let, $f(n) = 3^{2n-1} + 2^{n+1}$

$\therefore$ we have, $f(1) = 3 + 4 = 7$

$\therefore$ The result is true for $n = 1$

Let, the result is true for $n = k$

$\therefore f(k) = 3^{2k-1} + 2^{k+1}$ is divisible by 7.

Now we have, $f(k+1) - 2f(k) = 3^{2k+1} + 2^{k+2} - 2(3^{2k-1} + 2^{k+1})$

$= 3^{2k+1} + \cancel{2^{k+2}} - 2 \cdot 3^{2k-1} - \cancel{2 \cdot 2^{k+1}}$

$= 3^{2k-1}(3^2 - 2) = 7 \cdot 3^{2k-1}$

Then $f(k+1) = 2f(k) + 7p$ $[p = 3^{2k-1}]$

$\therefore$ $f(k+1)$ is divisible by 7.

$\therefore$ By the principle of induction $f(n)$ is divisible by 7 for all natural values of n.

1) Show that if a/b and b/c then a/c.

$\Rightarrow$ Since, a/b

$\therefore$ there exists an integer m such that $b = ma$

again since, b/c, there exists an integer n such that $c = nb$

$\therefore$ we have,

$$c = nb$$
$$= nma$$
$$= pa, \quad p = mn \in \mathbb{Z}$$

$\therefore a/c$..

2) Show that $\gcd(a, a+2) = 1$ or 2 $\forall a \in \mathbb{Z}$

$\Rightarrow$ Let, $\gcd(a, a+2) = d$

$\therefore d/a$ and $d/a+2$

$\therefore d \mid \{ax + (a+2)y\}$ $\forall x, y \in \mathbb{Z}$

~~$\therefore d \mid \{$~~ Choosing $x = -1$ and $y = 1$ we have,

$d \mid \{-a + (a+2)1\}$

$\Rightarrow d \mid 2$

$\Rightarrow d = 1$ or 2

3) If p is a prime and if $a, b \in \mathbb{Z}$ s.t p/ab then show that p/a or p/b.

$\Rightarrow$ If p/a then the result is proved.

~~Let p does not div~~

Let, $p \nmid a$

$\therefore \gcd(p, a) = 1$

$\therefore$ There exists integers x and y such that $px + ay = 1$ — (i)

Now, we have, $p \mid pb$

and given that p/ab

$\therefore p \mid (pbx + aby)$

$\Rightarrow p \mid b(px + ay)$

$\Rightarrow p/b$ [by (i)]

4) ~~If~~ p is a Prime and $1 \leq a < p$ then show that p is Prime to a.

$\Rightarrow$ Let, $\gcd(p, a) = d$

$\therefore$ d/p and d/a

Now, $d/p \Rightarrow d = 1$ or p

Since, $a < p$

$\therefore$ $d \neq p$

$\therefore$ $d = 1$

$\therefore$ p is prime to a.

5) If a is prime to b then Prove that (a^2+b^2) is Prime to a^2b^2.

6) If a is Prime to b then prove that $(a+b)$ is Prime to ab.

~~6)~~ If a is Prime to b then Prove that

7) If $p \geq q \geq 5$ and p, q are both Primes then prove that $24 / (p^2 - q^2)$

8) ~~If p be~~ Let, p be a prime and a be a positive integer then prove that a^n is divisible by p if and only if a is divisible by p.

9) Prove that no prime factor of (n^2+1) can be of the form $4m-1$, where m is integer.

5) $\because$ a is Prime to b

$\therefore$ $\gcd(a, b) = 1$

$\therefore$ There exists two integers x, y such that $ax + by = 1$ —(i)

Now, let, $\gcd(a^2+b^2, a^2b^2) = d$

$\therefore$ d / a^2+b^2 and d / a^2b^2 $\Rightarrow$ $a^2+b^2 = dm$ and $a^2b^2 = nd$; $n, m \in \mathbb{Z}$

Now, there exists two integers u and v such that $(a^2+b^2)u + a^2b^2v = d$

or, $1 = \frac{a^2+b^2}{d}u + \frac{a^2b^2}{d}v$

$= mu + nv$

$\therefore$ (a^2+b^2) and a^2b^2 are prime to each other. $\because \gcd(m, n) = 1$

Relation

Cartesian Product:- The Cartesian products of two sets A and B is denoted by $A \times B$ and defined to be $A \times B = \{(x,y) : x \in A \text{ and } y \in B\}$

For example, consider the sets $A = \{a, b, c\}$ and $B = \{1, 2\}$

Then $A \times B = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$

Note:- $o(A \times B) = o(A) \times o(B)$

Relation set:- Let, A be a set and ρ be a subset of $A \times A$ such that $(a, b) \in \rho$ if and only if a and b satisfie a particular relation.

For example, consider $A = \{1, 2, 3\}$

Here $A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

Let, ρ be a subset of $A \times A$ if and only if $(a, b) \in \rho \Rightarrow a < b$.

Then, $\rho = \{(1,2), (1,3), (2,3)\}$

The set ρ is called a relation set on A.

Note:- If $(a,b) \in \rho$ then we write symbolically $a \rho b$.

Types of relation:- i) **Reflexive relation:-** Let, A be a set and ρ be the relation defined in it. ρ is said to be reflexive if $a \rho a$ i.e. $(a,a) \in \rho$, $\forall a \in A$

ii) **Symmetric relation:-** Let, A be a set and ρ be a relation in it. ρ is said to be symmetric if $a \rho b \Rightarrow b \rho a$.

iii) **Antisymmetric relation:-** Let, A be a set and ρ be a relation in it. ρ is said to be antisymmetric if $a \rho b$ and $b \rho a \Rightarrow a = b$

iv) **Transitive relation:-** Let, A be a set and ρ be a relation defined in it. ρ is said to be transitive if $a \rho b$ and $b \rho c \Rightarrow a \rho c$

Equivalence relation:- Let, A be a set and ρ be a relation defined in it. ρ is said to be equivalence relation if it is reflexive, symmetric and transitive.

1) Let, $S = \{a, b, c\}$ and $\rho = \{(a,a), (b,c), (c,b)\}$. Is ρ an equivalence relation? Justify your answer.

$\Rightarrow$ Since, "$a \rho a$" does not holds for $\forall a \in S$.

$\therefore$ ρ is not reflexive.

$\therefore$ ρ is not an equivalence relation.

2) A relation ρ is defined on N by $m \rho n$ iff "m is a divisor of n", $\forall m, n \in N$.
Prove that ρ is transitive but not symmetric.

$\Rightarrow$ Let, $a \rho b$ and $b \rho c$ hold.

Now, $a \rho b \Rightarrow a/b$ — (i)

and $b \rho c \Rightarrow b/c$ — (ii)

from (i) and (ii),

a/c

$\Rightarrow a \rho c$

$\therefore$ ρ is transitive.

clearly, $5 \rho 15$ holds but $15 \rho 5$ does not hold.

$\therefore$ ρ is not symmetric.

3) Give an example of a relation which is reflexive but not symmetric.

$\Rightarrow$ Let, ρ be a relation on the sets of natural number N.
defined by $a \rho b$ iff $a \geq b$, $a, b \in N$

clearly, $a \rho a$ holds $\forall a \in N$

$\therefore$ ρ is reflexive.

clearly $7 \rho 5$ hold but $5 \rho 7$ does not holds.

$\therefore$ ρ is not symmetric.

4) On the set of complex number $\mathbb{C}$ a relation ρ is defined by $z_1 \rho z_2$ iff $z_1 z_2 \geq 0$. Determine wheather ρ is reflexive or symmetric or transitive.

$\Rightarrow$ Since, $i \cdot i = -1 \not\geq 0$ $\therefore$ $i \rho i$

$\therefore$ ρ is not reflexive.

we have, $z_1 \rho z_2 \Rightarrow z_1 z_2 \geq 0$

$\Rightarrow z_2 z_1 \geq 0$

$\Rightarrow z_2 \rho z_1$

$\therefore \rho$ is symmetric.

~~When~~ we have, $i \rho -i$ and $-i \rho i$ hold but $i \rho i$

$\therefore \rho$ is not transitive.

Home - Work

6) Since, a and b are prime to each other

$\therefore$ there exists two integers x and y such that $ax + by = 1$

Now, ~~$a(x+y) - y(a*b) = 1$~~

$\therefore a(x-y) + y(a+b) = 1$

$\because (x-y)$ and y are integers

$\therefore a$ and $(a+b)$ is prime to each other.

again, $b(y-x) + x(a+b) = 1$

$\because (y-x)$ and x are integers

$\therefore b$ and $(a+b)$ is prime to each other.

$\therefore (a+b)$ is prime to ab.

5) Since, a and b are prime to each other.

Since, a and b are prime to each other such that $ax + by = 1$

$\therefore$ there exists two integers x and y

~~Now, $ax + by = 1$~~

~~or, $(ax + by)^2 = 1$~~

~~or, $a^2x^2 + b^2y^2 + 2abxy = 1$~~

~~or, $(a^2+b^2)x^2 + y^2(b^2) - x^2b^2 - 2abxy = 1$~~

~~or, $(a^2+b^2)x^2 + b^2y^2 - xb(xb + 2ay) = 1$~~

~~or, (a^2+b^2) and b^2 are prime to each other.~~

~~If (a^2+b^2) and b^2~~

~~$\therefore (a^2+b^2)m + b^2n = 1$~~

~~or,~~

$\because (a+b)$ is Prime to ab

$\therefore (a+b)u + abv = 1 \quad , u, v \in \mathbb{Z}$

or, $(a+b)u = 1 - abv$

or, $(a^2+b^2+2ab)u^2 = 1 + a^2b^2v^2 - 2abv$

or, $\cancel{(a^2+b^2)u^2 + 2ab(u+v) - a^2b^2v^2} = 1$

or, $\cancel{(a^2+b^2)u^2 + a^2b^2v - 2a^2b^2v + 2ab(u+v)} = 1$

or, $\cancel{(a^2+b^2)}u^2 + a^2b^2v - 2ab(u+v-abv) = 1$

or, $(a^2+b^2)u^2 + 2abu^2 + 2abv - a^2b^2v^2 = 1$

or, $(a^2+b^2)\cancel{u}^2 + ab($

$a^2b^2v^2 = [1-(a+b)u]^2 = 1 + (a+b)^2u^2 - 2u(a+b)$

$= 1 + (a^2+b^2+2ab)u^2 - 2au - 2bu$

$= 1 + a^2u^2 + b^2u^2 + 2abu^2 - 2au - 2bu$

or, $a^2b^2v^2 + 2au + 2bu - a^2u^2 - b^2u^2 - 2abu^2 = 1$

or, $a^2b^2v^2 + 2u(a+b) - u^2(a^2+b^2) - 2abu^2 = 1$

or, $a^2b^2v^2 + u^2(a^2+b^2) - 2u^2(a^2+b^2) + 2u(a+b) - 2abu^2 = 1$

or, $a^2b^2v^2 + u^2(a^2+b^2) - 2u(a^2u + b^2u - a - b + abu) = 1$

or, $a^2b^2v^2 + u^2(a^2+b^2) - 2u[u(a+b)^2 - (a+b)] = 1$

or, $a^2b^2v^2 + u^2(a^2+b^2) - 2u[(a+b)(au+ub-1)] = 1$.

Functions

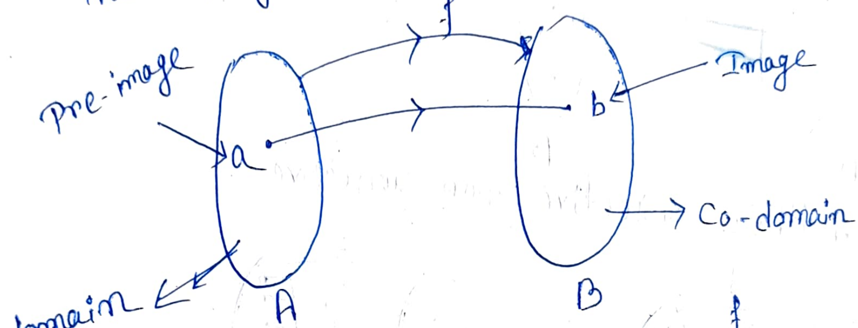
2.10.18

(1) **Definition :-** Let, A and B be two non empty sets (not necessarily different). A map or mapping or function f from A into B is defined to be a subset of $A \times B$ [i.e. f is a relation from A to B] so that the following properties are satisfied —

i) To each $a \in A$ there corresponds some $b \in B$ such that $(a, b) \in f$

and ii) $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$.

The mapping f is symbolically denoted by $f : A \to B$



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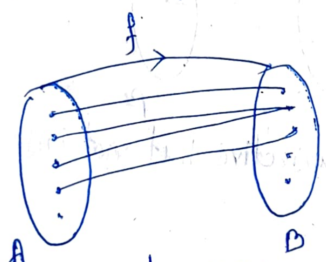
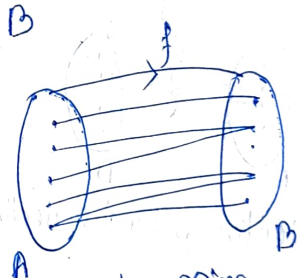
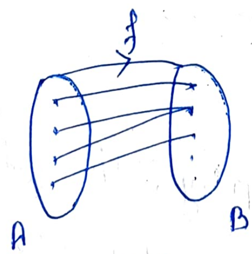


not mapping



not-mapping



mapping

(3) **Types of mapping :-**

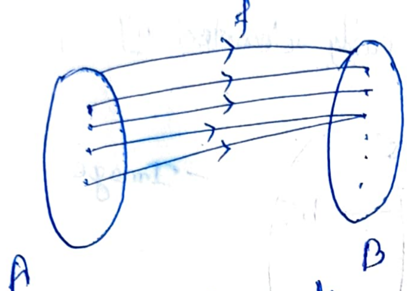
1) Injective map / one-to-one mapping :- A mapping $f : A \to B$ is said to be injective if for $x_1, x_2 \in A$, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

i.e. $f(x_1) = f(x_2)$

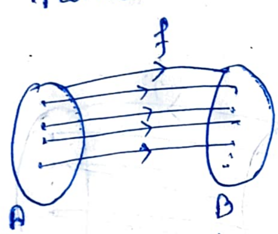
$\Rightarrow x_1 = x_2$

2) Surjective or onto mapping :- A mapping $f: A \to B$ is said to be surjective or onto if for every element $b \in B$ there exists some elements $a \in A$ such that $f(a) = b$.

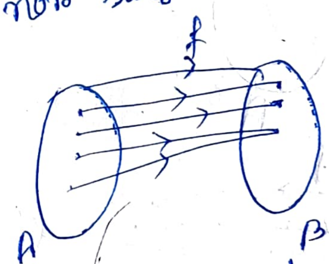
3) Bijective / one-one mapping :- A mapping $f: A \to B$ is said to be bijective if it is injective and surjective.



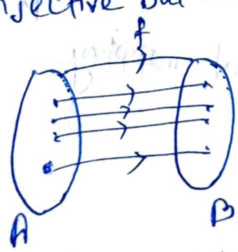
neither injective nor surjective



Injective but not surjective



Surjective but not injective



Bijective mapping

Range set :- Let, $f: A \to B$ be a mapping. The set of elements of the image of the elements of A is called range set and it's denoted by $f(A)$. Clearly, ~~f(A) is~~ $f(A) \subseteq B$.

[Note :- If $f: A \to B$ be an onto mapping then $f(A) = B$.]

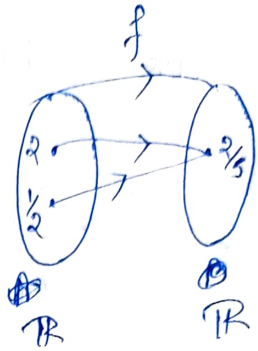
Theorem :- If $f: A \to B$ be a bijective mapping then the cardinallities of A and B are same. H.W.

1) Show that the mapping $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x}{x^2+1}$, $\forall x \in \mathbb{R}$ is neither injective nor surjective.

$\Rightarrow$ We have,

$$f(2) = \frac{2}{5}$$

and $f\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{1+\frac{1}{4}} = \frac{2}{5}$



$\therefore$ we have, $2 \neq \frac{1}{2}$

but, $f(2) = f\left(\frac{1}{2}\right)$

$\therefore$ The mapping is not injective.

The element $1 \in \mathbb{R}$ has no pre-image for if a be the pre-image of 1 then,

$$\frac{a}{a^2+1} = 1$$

or, $a^2 - a + 1 = 0$

or, $a = \frac{1 \pm \sqrt{(-1)^2 - 4}}{2}$

$= \frac{1 \pm i\sqrt{3}}{2} \notin \mathbb{R}$

$\therefore f$ is not surjective.

2) Show that the mapping $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{x}{1+|x|}$, $\forall x \in \mathbb{R}$ is one to one.

$\Rightarrow$ We have,

$$f(x) = \frac{x}{1+|x|}$$

$$= \begin{cases} \frac{x}{1+x}, & x \geq 0 \\ \frac{x}{1-x}, & x < 0 \end{cases}$$

<u>Case I :-</u> Let, $x \geq 0$

In this case we have,

$f(x_1) = f(x_2)$

$\Rightarrow \frac{x_1}{1+x_1} = \frac{x_2}{1+x_2}$

$\Rightarrow x_1 + x_1x_2 = x_2 + x_1x_2$

$\Rightarrow x_1 = x_2$

2) ∴ In this case $f(x)$ is one-to-one.

Case II :- Let $x < 0$

$f(x)$ in this case,

$$f(x_1) = f(x_2)$$

$$\Rightarrow \frac{x_1}{1-x_1} = \frac{x_2}{1-x_2}$$

$$\Rightarrow x_1 - x_1x_2 = x_2 - x_1x_2$$

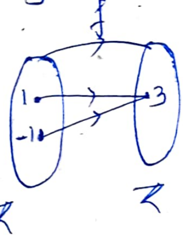
$$\Rightarrow x_1 = x_2$$

∴ In this case $f(x)$ is one-to-one.

Combining Cases I and II $f(x)$ is one-to-one.

3) Let, Z be the set of all integers. Is the mapping $f: Z \to Z$ defined by $f(x) = x^2+2$, $\forall x \in Z$ a one-to-one mapping? Justify your answer.

⇒ We have, $f(1) = 3 = f(-1)$



∴ $1 \neq -1$

But $f(1) = f(-1)$

∴ f is not one-to-one.

4) Show wheather the mapping $f: Z \to Z^+$ defined by $f(x) = |x|$, $\forall x \in Z$ and Z is the set of all integers Is onto or not?

⇒ Let, b be any element of the Codomain set.

Let, a be the pre-image of b under f.

∴ $f(a) = b$

⇒ $|a| = b$

⇒ $a = \pm b$

⇒ $a = b, -b$

∴ b and $-b$ are the Preimage's of B

∴ f is onto.

5) Give an example with justification, of a mapping which is surjective but not injective.

$\Rightarrow$ $f(x) = |x|$, $\forall x \in Z$

6) Show that the mapping $f: C \to C - \{0\}$, where C is the set of complex numbers defined by $f(z) = e^{z}$, $\forall z \in C$ is onto but not one-to-one.

$\Rightarrow$ Let, z be any element of the co-domain set.

Let, a be the pre-image of z under f.

$\therefore f(a) = z$

or, $e^{a} = z$

or, $a = \log z \in C$

$\therefore \log z$ is the preimage of z under f

$\therefore f$ is onto.

Let, $z_1 = \pi i$ and $z_2 = 3\pi i$

we have, $f(z_1) = e^{\pi i} = \cos\pi + i\sin\pi = -1$

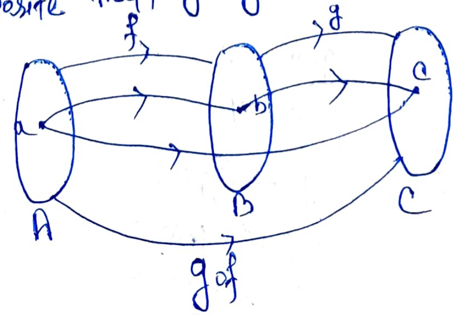
and $f(z_2) = e^{3\pi i} = \cos 3\pi + i\sin 3\pi = -1$

Thus, $z_1 \neq z_2$

But $f(z_1) = f(z_2)$

$\therefore f$ is not one-to-one.

Composite mapping :- Let, $f: A \to B$ and $g: B \to C$ be two mappings. The composite mapping $g \circ f: A \to C$ is defined by $(g \circ f)(x) = g(f(x))$



1) Two mappings $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ are defined as $f(x) = x^2+1, \forall x \in \mathbb{R}$ and $g(x) = x, \forall x \in \mathbb{R}$. find the composite mapping $f \circ g$ and $g \circ f$.

$\Rightarrow$ We have,

$$f \circ g = f(g(x)) = f(x) = x^2+1$$

and $g \circ f = g(f(x)) = g(x^2+1) = x^2+1$

[Note:— In general $f \circ g \neq g \circ f$]

2) Let, $f: S \to T$ and $g: T \to U$, Show that $g \circ f$ is one-to-one if each of g and f is one to one.

$\Rightarrow$ Let, $h = g \circ f$

We have, $h(x_1) = h(x_2)$, $x_1, x_2 \in S$ Let, $y_1 = f(x_1)$ and $y_2 = f(x_2)$.

$\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow g(y_1) = g(y_2)$

$\Rightarrow y_1 = y_2$ [since, g is one-to-one]

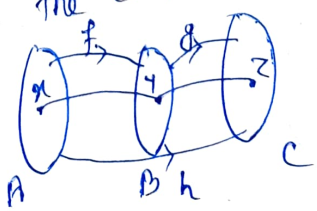
$\Rightarrow f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2$ [since, f is one-to-one]

$\therefore g \circ f$ is one-to-one.

3) Let, $f: A \to B$ and $g: B \to C$ be both bijective mappings, Show that the composite mapping $g \circ f: A \to C$ is bijective.

$\Rightarrow$



$g(y) = z$

$f(x) = y$

$\therefore g(f(x)) = z$

$(g \circ f)(x) = z$

$h(x) = z$

4) Let, $f: A \to B$ and $g: B \to A$ be two mappings such that $g \circ f = I_A$, the identity mapping on A. Show that f is one-to-one and g is onto.

$\Rightarrow$ Let, $h = g \circ f$

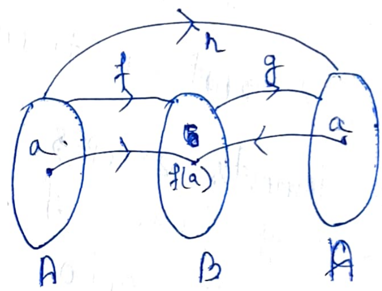
Since, h is identity mapping on A

$\therefore h(x) = x, \quad \forall x \in A$

~~Now Let~~, for $x_1, x_2 \in A$ we have,

$$f(x_1) = f(x_2)$$
$$\Rightarrow g(f(x_1)) = g(f(x_2))$$
$$\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$$
$$\Rightarrow h(x_1) = h(x_2)$$
$$\Rightarrow x_1 = x_2$$

$\therefore f$ is one-to-one



Let, a be any element of A.

$\therefore f(a) \in B$

Now we have, $g(f(a))$
$$= (g \circ f)(a)$$
$$= h(a)$$
$$= a$$

This shows that $f(a)$ is the preimage of A under g.

$\therefore g$ is onto.

Inverse of a mapping :—

1) If $f: \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $f(x) = x^2 + 3$, where $\mathbb{R}^+$ is the set of all positive real numbers, then find $f^{-1}(7)$

$\Rightarrow$ Let, $f^{-1}(7) = x$

$\therefore 7 = f(x) = x^2 + 3$

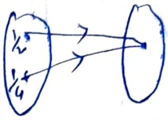
or, $x = \pm 2$

$\therefore x = 2 \quad [\because f: \mathbb{R}^{+} \to \mathbb{R}^{+}]$

2) A mapping $f: \mathbb{R} \to \mathbb{R}$ defined as follows $f(x) = \begin{cases} 1, & x \text{ is rational} \\ -1, & x \text{ is irrational} \end{cases}$
Find $f(\sqrt{7})$. Is the mapping one-to-one or onto?
Find the range set of f.

$\Rightarrow$ Since, $\sqrt{7}$ is irrational

$\therefore f(\sqrt{7}) = -1$



We have, $f\left(\frac{1}{2}\right) = 1$

$f\left(\frac{1}{4}\right) = 1$

$\because \frac{1}{2} \neq \frac{1}{4}$

but $f\left(\frac{1}{2}\right) = f\left(\frac{1}{4}\right)$

$\therefore$ f is not one-to-one

The elements excepts '-1' and '1' of '$\mathbb{R}$' has no pre-image under f.

$\therefore$ f is not onto.

$\therefore$ The range set of f $\{-1, 1\}$